MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

VOL. LXII.

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WITH MANY

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AN APPENDIX.

EDITED BY

W. J. C. MILLER, B.A.,

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Solved Questions.

ERRATA IN VOL. LXI.

Page 39, line 7, *for* Montfort *read* Montmort, and add at end of solution references to Todhunter's *Theory of Probability*, Sections 160, 161, 281, 288, 627, 979.

Page 35, in (10) of Quest. 11683, the first figure should be 3, not 2.

„ 99, last line, *for* $r, r+1$ *read* $r+1, r$.

„ 100, Art. 4, line 4, *for* ${}^nQ_{r+1}$ *read* ${}_nQ_{r+1}$, and *for* ${}_nR_r$ *read* ${}_nQ_r$.

„ 100, Art. 6, line 8, *for* Hence *read* Then.

„ 101, Art. 6, top line of page, *for* $(1-Jx)$ *read* $(1+Jx)$.

„ 101, Art. 7, line 2, *after* ${}_nQ_n$ *insert* = ;
and *for* $-(n-1)(1-J)^{n-2}J^2$ read $-(n-1)(1-J)^{n-2}J$;
and in line 3, *for* $-(n-2)(1-J)^{n-3}J^2$ *read* $-(n-2)(1-J)^{n-3}J$.

„ 101, Art. 8, last line, *for* e^- *read* e^{-2}.

ERRATA IN VOL. LXII.

Pp. 58–9, lines 3, 4 of Quest. 12424, *for* $\text{cosec}^2\frac{1}{2}B$, $\text{cosec}^2\frac{1}{2}C$, *read* $\text{cosec}^2\frac{1}{2}A$, and *for* poles *read* polars.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

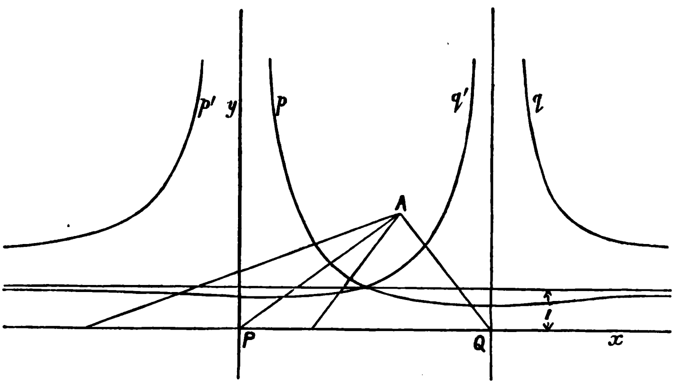
WITH ADDITIONAL PAPERS AND SOLUTIONS.

6466. (Rev. C. Taylor, D.D.) — The bisectors of the vertical angle A of a triangle meet the base in P, Q; trace the variations of magnitude in AO : OP and AO : OQ as O moves along the base, and apply the result to prove that a conic is concave to its axis.

Solution by H. J. Woodall, A.R.C.S.

Let P be origin of rectangular coordinates, PQ the axis of x, and A (h, k). Then the locus of

$$y = \text{AO} : \text{OP}, \quad \text{is} \quad xy = \{(x-h)^2+k^2\}^{\frac{1}{2}},$$



which is part of a quartic. [In the figure the curves give the arithmetical value of y only. Also those relating to y = AO : OP are marked p, p', while q, q' relate to Q.]

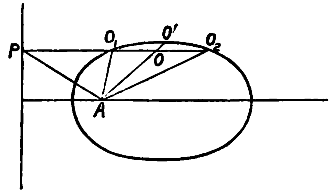
Take a value (h) of y = AO : OP, and draw $y-h = 0$. This will cut pp' in two points, K_1, K_2 only (corresponding to O_1, O_2 on the axis).

K_1, K_2 will be on the same branch (p), and on the same side of the y-axis if h be < 1.

If $h > 1$, K_1, K_2 will be one on each branch, and on opposite sides of Py.
If $h = 1$, one of these points will be at infinity.

Consider the curve as commencing at infinity on the y-axis, passing along the branch p to infinity on the x-axis, and returning along the branch p'. If then we draw $y = h$ to cut the curve at K_1K_2, P may be considered as lying outside the part O_1O_2 of the x-axis. In which case we may say that, for a point O in PQ, $AO : OP \lessgtr AO_1 : O_1P$ (*i.e.* $\lessgtr h$), according as O is inside or outside of O_1O_2.

In a conic draw a chord O_1O_2 parallel to the x-axis to meet the y-axis (directrix) at P. Join AP, AO_1, AO_2. Take any point O on O_1O_2. Then



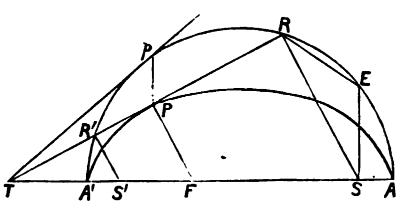
$$AO : OP \lessgtr AO_1 : O_1P,$$

according as O lies inside or outside of O_1O_2. Hence to find a new point O′ so that
$$AO' : O'P' = AO_1 : O_1P,$$
we must lengthen or shorten AO according as O lies inside or outside of O_1O_2. Hence the curve is concave to the axis.

12368. (J. Macleod, M.A.)—From a point T on the major axis of an ellipse, tangents TP, Tp are drawn to the ellipse and auxiliary circle, respectively; TP is produced to meet the circle in R, and PF perpendicular to TP meets the major axis in F. Show that (1), S and S′ being the foci, $Tp^2 : TR^2 = SF : FS'$; and (2) if TP bisects the angle pTS, and S′E, meeting the circle in E, is perpendicular to the major axis,
$$\angle S'ER = RpT.$$

Solution by W. J. Dobbs, M.A., Professor Bhattacharya, *and others.*

Pp is perpendicular to the axis, and TR′PR is harmonic.



Also, R and R′ are the feet of the perpendiculars from S, and
$$RS . R'S' = BC^2 = AS . SA' = SE^2.$$

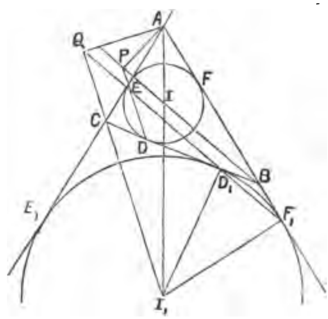
Now, $Tp^2 : TR^2$
$$= TR' . TR : TR^2 = TR' : TR = PR : RP = SF : FS'.$$

Again, $RS^2 : SE^2 = RS^2 : RS \,.\, R'S' = RS : R'S' = TR : TR'$
$= TR^2 : Tp^2$ (as above); $\therefore$ $RS : SE = TR : Tp$.

Also, $\angle RSE = RTS = RTp$, if TP bisects $\angle pTS$; therefore, in this case, triangles RSE, RTp are similar; and therefore SER = TpR.

12352. (C. E. Hillyer, M.A.)—If, from the vertices of a triangle, straight lines be drawn perpendicular, respectively, to the internal and external bisectors of the other two angles, the feet of these 12 perpendiculars lie on the sides of the "in-triangle" and the three "ex-triangles" of the original triangle; the "in-triangle" denoting the triangle formed by joining the points of contact of the inscribed circle, and an "ex-triangle" that formed by joining the points of contact of an escribed circle.

Solution by W. J. Dobbs, M.A.; H. W. Curjel, B.A.; *and others.*



Let DEF be the in-triangle of ΔABC and I the in-centre; and let P be the foot of the perpendicular from A on BI.

Then IEPA are concyclic.

$\therefore$ $\angle AEP = \angle AIP = \frac{1}{2}(A+B)$
$= \angle DEC$;

therefore P lies in DE. Similarly, the feet of the other perpendiculars from the vertices on the internal bisectors lie in the sides of ΔDEF.

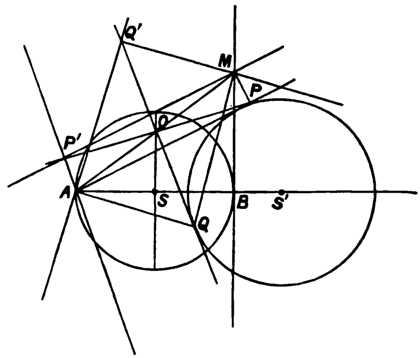
Again, let I_1 be the ex-centre within $\angle A$, and $D_1E_1F_1$ the corresponding ex-triangle, and Q the foot of the perpendicular from A on the external bisector of the $\angle C$. Then A, Q, I_1, F_1 are concyclic;

$$\therefore \quad \angle QF_1A = \angle QI_1A = \tfrac{1}{2}B = \angle D_1F_1A;$$

therefore Q lies on D_1F_1, and similarly the other feet of perpendiculars from the vertices on the external bisectors lie on the sides of the ex-triangles. Similarly, Q lies on D_2F_2 and P on D_2E_2, &c.

8206. (Editor.)—A circle S has AB for diameter, another circle S′ has its centre on AB and cuts S at right angles; from any point O on the diameter of S, which is at right angles to AB, are drawn OP, OQ tangents to S′; AP′, AQ′ are drawn at right angles to AP, AQ, respectively, and meeting OP, OQ in P′, Q′; prove that the straight lines drawn through P′, Q′ parallel respectively to AP, AQ will intersect in a point lying on AO, and also on the tangent to S at B.

Solution by Professors Zerr, Mukhopadhyay, *and others*.

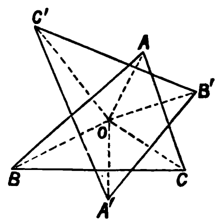


Let r, r_1 be the radii of S, S′; call SS′ a, and let SO $= k$.

Produce AO to M; then, since AS = SB, AO = OM; also, since $a^2 - r^2 = r_1^2$, from the fact that the circles cut at right angles, we easily get that $AO = OP = OQ = (r^2 + k^2)^{\frac{1}{2}}$. Also produce PO to P making OP′ = PO, produce QO to Q′ making OQ′ = QO. Join AP′, AQ′, Q′M, P′M, QM, PM. Since the diagonals of the quadrilaterals AP′MP, AQ′MQ bisect each other and are equal, each is a rectangle; therefore AQ′ is perpendicular to AQ, Q′M, and AP′ is perpendicular to AP, P′M. Therefore Q′M and P′M intersect at a point M which lies on AO, and also on BM the tangent to S at B.

12276. (R. Tucker, M.A.)—If the radical centre of three circles is the orthocentre of the triangle formed by joining their centres, it is also of the triangle formed by the polars of the radical centre. Examine cases of common in-centres, &c.

Solution by W. J. Dobbs, M.A.; Morgan Brierley; *and others*.



It is well known that the polars of a point with respect to a system of co-axial circles are concurrent, and, if the point be on the axis of the system, so is the point of concurrency.

Now, let O be the radical centre of the three circles, centres A, B, C, and let A′B′C′ be the triangle formed by the polars of O with respect to the circles.

Then, from the above, A′, B′, C′ are points on the radical axes. Hence OA′, OB′, OC′ are perpendicular to the sides of triangle ABC.

Thus the triangles ABC, A′B′C′ possess the property that the joins of O to the angular points of one are respectively perpendicular to the sides of the other.

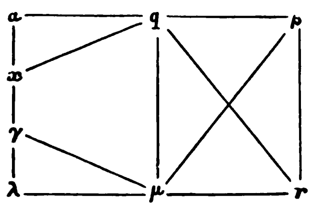
If O be the orthocentre of one, AOA′, BOB′, COC′ are straight lines, and O is therefore the orthocentre of the other.

If O be the in-centre of one, it is easily seen that O is the circumcentre of the other.

12349. (D. Biddle.)—In order to solve $x^3+qx+r=0$, when Cardan's method is inoperative because q is a minus quantity and $(\frac{1}{27}q^3+\frac{1}{4}r^2)$ also negative, take $\alpha^3+q\alpha+\beta=0$ and $\lambda^3+\mu\lambda+r=0$ such that $\frac{1}{27}q^3+\frac{1}{4}\beta^2=0$ and $\frac{1}{27}\mu^3+\frac{1}{4}r^2=0$, whence $\beta=-2(-\frac{1}{3}q)^{\frac{3}{2}}$, $\alpha=2(-\frac{1}{3}q)^{\frac{1}{2}}$, $\mu=-3(\frac{1}{2}r)^{\frac{2}{3}}$, $\lambda=2(-\frac{1}{2}r)^{\frac{1}{3}}$. Note that x lies between α and λ, and is nearly given by $(\frac{2}{3}\alpha^3+\frac{1}{3}\lambda^3)^{\frac{1}{3}}$. With the values arrived at, however, take a third subsidiary equation, $\gamma^3+\mu\gamma+\beta=0$, and, finding γ by Cardan's method, which never fails here, prove that $\gamma-\lambda : x-\gamma = x-\gamma : \alpha-x$ nearly, and $\frac{1}{2}\left[\lambda+\gamma\pm\left\{(\lambda+\gamma)^2-4\gamma^2+4\alpha(\gamma-\lambda)\right\}^{\frac{1}{2}}\right]$ is a close approximation to a real value of x. This holds good more particularly if the coefficients represented by q and r be equalized by taking $y=\frac{q}{r}x$, whence we get $y^3+\frac{q^3}{r^2}y+\frac{q^3}{r^2}=0$.

Solution by the Proposer.

We have here to consider the partial deformation of an equation, and the variation of the unknown quantity, whilst one of the coefficients remains fixed. This can best be illustrated by diagram :



..................(A).

Let us first take $\alpha^3+q\alpha+\beta=0$ as the equation to be deformed, whilst q remains fixed. β gradually changes to r, and α to x, and we get $x^3+qx+r=0$, the equation to be solved. Then, if r remain fixed, and q change to μ, $\lambda^3+\mu\lambda+r=0$ appears. We now reverse the process, and, keeping μ fixed, allow r to pass back to β, when we get $\gamma^3+\mu\gamma+\beta=0$. Then β being fixed, and μ passing to q, we reach our starting point, $\alpha^3+q\alpha+\beta=0$.

Let $z=$ the unknown quantity as it varies from α through x to λ, and back again from λ through γ to α ; and let Q, R represent the moveable coefficients as they vary, the former between q and μ, the latter between r and β. During each separate stage of the deformation, only one of them is in operation, wherefore we have these four equations

$$z^3+qz+R=0,\quad z^3+Qz+r=0 \quad\text{......................(B, C)},$$

$$z^3+\mu z+R=0,\quad z^3+Qz+\beta=0 \quad\text{....................(D, E)}.$$

In all four cases, the variant Q or R proceeds from one known limit to another, and each varies between limits of its own which are constant.

Now, assuming $\gamma-\lambda : x-\gamma = x-\gamma : \alpha-x$ to be absolutely, and not only (as in the question), approximately true, we get

$$x-\lambda = (\gamma-\lambda)^{\frac{1}{2}}\left\{(\gamma-\lambda)^{\frac{1}{2}}+(\alpha-x)^{\frac{1}{2}}\right\},$$

and $$\alpha-\gamma = (\alpha-x)^{\frac{1}{2}}\left\{(\gamma-\lambda)^{\frac{1}{2}}+(\alpha-x)^{\frac{1}{2}}\right\},$$

whence also $$\gamma-\lambda : \alpha-x = (x-\lambda)^2 : (\alpha-\gamma)^2 \quad\text{.....................(F)}.$$

During the variation represented by (B), z passes through $\alpha-x$, during (C) through $x-\lambda$, during (D) through $\gamma-\lambda$, and during (E) through $\alpha-\gamma$.

In (B), $dR/dz = -(3z^2+q)$; in (C), $dQ/dz = -(2z-r/z^2)$; in (D), $dR/dz = -(3z^2+\mu)$; in (E), $dQ/dz = -(2z-\beta/z^2)$. From the values given in the question, we have

$$q = -\tfrac{3}{4}\alpha^2, \quad \mu = -\tfrac{3}{4}\lambda^2, \quad r = -\tfrac{1}{4}\lambda^3, \quad \beta = -\tfrac{1}{4}\alpha^3;$$

therefore dR/dz, in (B), $: dR/dz$, in (D), $= 3z^2-\frac{3}{4}\alpha^2 : 3z^2-\frac{3}{4}\lambda^2$, and dQ/dz, in (C), $: dQ/dz$, in (E), $= 2z+\lambda^3/4z^2 : 2z+\alpha^3/4z^2$. Consequently, bearing in mind that z has different values in the different terms, we have roughly $z^2-\frac{1}{4}\lambda^2 : z^2-\frac{1}{4}\alpha^2$, and $z+(\lambda/2z)^3 z : z+(\alpha/2z)^3 z$. These become

$$z^2\left\{1-(\lambda/2z)^2\right\} : z^2\left\{1-(\alpha/2z)^2\right\}, \text{ and } z\left\{1+(\lambda/2z)^3\right\} : z\left\{1+(\alpha/2z)^3\right\}.$$

Taking the first and third of these, and the second and fourth, as being respectively most allied to each other (the z of one partially coinciding with the z in the other), we get

$$z^2\left\{1-(\lambda/2z)^2\right\} : z\left\{1+(\lambda/2z)^3\right\}, \text{ and } z^2\left\{1-(\alpha/2z)^2\right\} : z\left\{1+(\alpha/2z)^3\right\}.$$

These, in succession, become

$$z^2\left\{1-\lambda/2z\right\} : z\left\{1-\lambda/2z+(\lambda/2z)^2\right\}, \text{and } z^2\left\{1-\alpha/2z\right\} : z\left\{1-\alpha/2z+(\alpha/2z)^2\right\};$$

also $$z^2/\left\{1-\lambda/2z+(\lambda/2z)^2\right\} : z/\left\{1-\lambda/2z\right\},$$

and $$z^2/\left\{1-\alpha/2z+(\alpha/2z)^2\right\} : z/\left\{1-\alpha/2z\right\}.$$

Wherefore, considering the slightly different values of z in the several terms of the respective ratios, we can see that (F) is an approximation to the truth; for $z^2/\left\{1-\lambda/2z+(\lambda/2z)^2\right\}$ bears about the same ratio to the square of $z/\left\{1-\lambda/2z\right\}$ that $z^2/\left\{1-\alpha/2z+(\alpha/2z)^2\right\}$ does to the square of $z/\left\{1-\alpha/2z\right\}$. Of course, in each stage we have taken only the mean dz, and the mean dR or dQ. If we attempt integration, our only results are impracticable logarithmic values.

12329. (Professor Neuberg.)—Soient A, B, C trois points en ligne droite, B situé entre A et C. On élève en A et C, d'un même côté de AC, les perpendiculaires $AA' = BC$, $CC' = AB$, et en B, de l'autre côté de AC, la perpendiculaire $BB' = AC$. Démontrer: (1) que l'angle de Brocard du triangle A'B'C' est égal à arc cot 2; (2) que les centres des

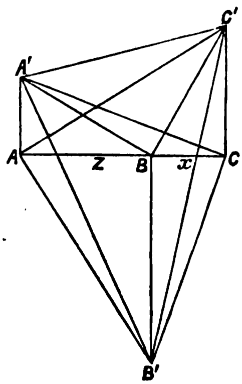
carrés construits intérieurement sur les côtés du triangle A′B′C′ sont en ligne droite.

Solution by C. Morgan, M.A.; H. W. Curjel, B.A.; *and others.*

Since A′AB, BCC′ are two equal right-angled triangles,

$$A'B = BC',$$

and $\angle A'BC'$ is a right angle; therefore B is the centre of the square on A′C′; similarly, A and C are the centres of squares on B′C′, A′B′; therefore the centres of these squares are collinear.



Let $BC = x$, $AB = z$,

and the Brocard angle of $\triangle A'B'C' = \omega$.

Then area of $\triangle A'B'C' = (x+z)^2 - xz$

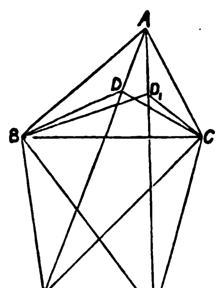
$$= x^2 + xz + z^2,$$

and $B'C'^2 + C'A'^2 + A'B'^2$

$$= 8(x^2 + xz + z^2);$$

therefore $\cot\omega = 8(x^2+xz+z^2)/4(x^2+xz+z^2) = 2.$

5240. (Professor Matz.)—A, B, C are given points; find the position of the straight line ADE, so that the quadrilateral BCED shall be given or a maximum.



Solution by Professor Lampe.

Let $BC = a$, $DE = b$, and the angle included by BC and DE $= \delta$; then area of quadrilateral BCED $= \frac{1}{2}ab\sin\delta$. (1) $\frac{1}{2}ab\sin\delta = q^2$, whence $\sin\delta = 2q^2/ab$, and this equation furnishes two positions of ADE, forming the equal sides of an isosceles triangle with BC. (2) $\sin\delta = 1$ gives the maximum area of the quadrilateral, or the perpendicular from A to BC furnishes the greatest quadrilateral BCD_1E_1.

12293. (Professor Haughton, F.R.S.) — The daily energy of the average diet of all the armies of Europe is estimated at 3694 foot-tons. The daily work done for this expenditure of food (including the work done in moving the man's own body) is estimated at 430·6 foot-tons: if man be regarded as a perfect heat-engine, whose upper temperature is

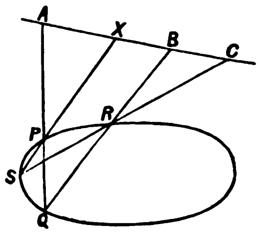
100° F. (blood-heat), calculate from the foregoing data the lower temperature at which the engine is worked.

Solution by H. J. Woodall, A.R.C.S.

Efficiency = work done/heat expended = 430·6/3694.

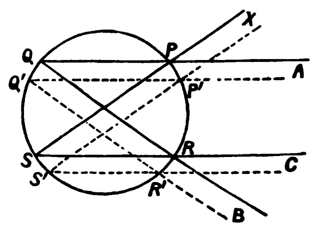
Efficiency = $(t_1 - t_0)/(460 + t_1) = (100 - x)/560 = 430{\cdot}6/3694$;

hence we have $x = 100 - 65 = 35°$ F.



12252. (W. J. Dobbs, B.A.) — If A, B, C are three fixed collinear points, P any point on a fixed conic, and AP, BQ, CR meet the conic in Q, R, S, prove that SP passes through a fixed point in ABC.

Solution by Professor Droz-Farny; R. F. Davis, M.A.; *and others.*



Projetons la figure orthogonalement sur un plan de manière à ce que les points d'intersection de ABC avec la conique deviennent les ombilics du plan.

La conique deviendra un cercle et les droites AP, BQ, CR auront des directions fixes. Il s'agit de démontrer que SP aura aussi une direction fixe. Soient AP′, BQ′, CR′ un second système de transversales respectivement parallèles à AP, BQ, CR; on aura

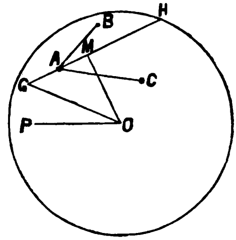
$$\text{arc } PP' = QQ' = RR' = SS'$$

donc S′P′ parallèle à SP ce qu'il fallait démontrer.

11931. (Professor Cavallin, M.A.)—Three points A, B, C are taken at random within a sphere; find the mean value of p^n, where p is the perpendicular from the centre of the sphere on the plane ABC and n any positive number. [The required mean value is unaltered when, instead of the point systems B, C, we take a system of chords (B′C′) with a density, perpendicular to B′C′, varying as $(B'C')^4$, all directions of the chords being equally probable; and from this altered form of the problem the solution is easily obtained.]

Solution by Professor ZERR.

Let GH be the diameter of the section of the sphere made by a plane through the three random points A, B, C; M its centre; O the centre of the sphere; OP a line such that AB is parallel to the plane MOP. Let $OG = r$, $MA = u$, $AB = v$, $AC = w$, $\angle GOM = \theta$, $\angle BAM = \phi$, $\angle CAM = \psi$, $\angle MOP = \lambda$, and the angle the plane POM makes with a fixed plane through $OP = \rho$.



An element of the sphere is,

at A, $\qquad r\sin\theta\, d\theta\,.\,2\pi u\, du$;

at B, $\qquad v^2 dv\, d\phi\, d\lambda$;

at C, $\qquad \sin(\phi+\psi)\sin\lambda\, w^2 dw\, d\psi\, d\rho$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of u, 0 and $r\sin\theta = u'$, and tripled; of ϕ, $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$; of ψ, $-\phi$ and $\frac{1}{2}\pi$; of λ, 0 and π; of ρ, 0 and 2π; of v, 0 and $2u\cos\phi = v'$; of w, 0 and $2u\cos\psi = w'$. $h = r\cos\theta$.

Hence, since the whole number of ways the three points can be taken is $(\frac{4}{3}\pi r^3)^3$, the required average is

$$\Delta = \frac{81}{64\pi^3 r^9}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{v'}\int_0^{w'} r^n\cos^n\theta\, r\sin\theta\, d\theta\, 2\pi u\, du \times \sin(\phi+\psi)\, d\phi\, d\psi \sin\lambda\, d\lambda\, d\rho\, v^2 dv\, w^2 dw$$

$$= \frac{27 r^{n-8}}{4\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{v'} u^4\cos^n\theta\sin\theta\sin(\phi+\psi)\cos^3\psi\sin\lambda \times d\theta\, du\, d\phi\, d\psi\, d\lambda\, d\rho\, v^2 dv$$

$$= \frac{18 r^{n-8}}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi} u^7\cos^n\theta\sin\theta\sin(\phi+\psi)\cos^3\phi\cos^3\psi\sin\lambda \times d\theta\, du\, d\phi\, d\psi\, d\lambda\, d\rho$$

$$= \frac{36 r^{n-8}}{\pi}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_0^{\pi}\cos^n\theta\sin\theta\, u^7\sin(\phi+\psi)\cos^3\phi\cos^3\psi\sin\lambda \times d\theta\, du\, d\phi\, d\psi\, d\lambda$$

$$= \frac{72 r^{n-8}}{\pi}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\cos^n\theta\sin\theta\, u^7\sin(\phi+\psi)\cos^3\phi\cos^3\psi \times d\theta\, du\, d\phi\, d\psi$$

$$= \frac{9 r^{n-8}}{\pi}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\left[3\left(\tfrac{1}{2}\pi+\phi\right)\sin\phi + 2\cos\phi + \sin^2\phi\cos\phi\right]\cos\ \theta\sin\theta\, u^7 \times \cos^3\phi\, d\theta\, du\, d\phi$$

$$= \frac{9\cdot 35 r^{n-8}}{32}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\cos^n\theta\sin\theta\, u^7\, d\theta\, du = \frac{9\cdot 35 r^n}{8\cdot 32}\int_0^{\frac{1}{2}\pi}\cos^n\theta\sin^9\theta\, d\theta$$

$$= \left(\frac{9}{8}\right)\left(\frac{35}{32}\right)\left\{\frac{1}{n+1}-\frac{4}{n+3}+\frac{6}{n+5}-\frac{4}{n+7}+\frac{1}{n+9}\right\} r^n.$$

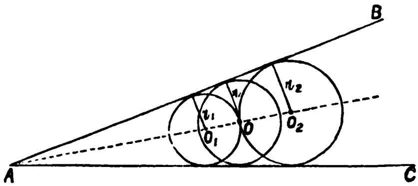
12239. (R. TUCKER, M.A.)—Three circles touch the straight lines AB, AC, two of them passing through the centre of the third; show that their radii are in harmonical progression.

Solution by J. H. Hooker, M.A.; R. H. W. Whapham; *and others.*

Let O_1, O, O_2 be the centres of the circles; r_1, r, r_2 their radii; and $AO = x$; then

$$\frac{x-r_1}{r_1} = \frac{x}{r} = \frac{x+r_2}{r_2};$$

$$\therefore \quad \frac{1}{r_1} + \frac{1}{r_2} = \frac{2}{r}.$$



1334. (Dr. Rutherford, F.R.A.S.)—A heavy uniform beam (AB) moves freely about a hinge at A, and an elastic string is attached to the extremity B, and fixed at a point C in the same horizontal line as A, at a distance (AC) equal to the length of the beam. The natural length of the elastic string is equal to half that of the beam, and its elasticity is such that the weight of the beam would stretch it to twice its natural length. Find the angle which the string makes with the horizon when the system is in equilibrium.

Solution by Professors Zerr, Bhattacharya, *and others.*

Let p be the perpendicular from A on BC, T = tension on the string, $AB = 2a$, $CB = 2l$,

$$ACB = ABC = \theta;$$

then $\quad p = 2a \sin\theta, \quad AD = a\cos(\pi - 2\theta);$

therefore $\quad T = -W\cos\theta;$

but $\quad \cos\theta = l/2a; \quad \therefore\ T = +(Wl)/2a.$

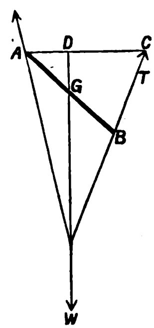
Also $\quad 2l = a(1 + T/\lambda) = a(1 + T/W),$

$$2l = a(1 + l/2a) = a + \tfrac{1}{2}l;$$

therefore $\quad l = \tfrac{2}{3}a,$

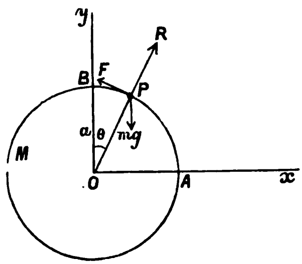
$$2l = BC = \tfrac{4}{3}a;$$

therefore $\quad \cos\theta = \tfrac{1}{3}, \quad \theta = \cos^{-1}(\tfrac{1}{3}).$



12290. (R. H. W. Whapham, M.A.)—A uniform sphere is capable of motion about a horizontal diameter. A small groove is cut in the sphere in the plane of the great Circle perpendicular to axis of rotation. In this groove, close to the highest point, is placed a small bead of mass two-fifths of the sphere. If the coefficient of friction between the bead and sphere be $\frac{1}{2}$, prove that before the bead begins to slide the sphere will have turned through an angle $\tan^{-1}\frac{3}{4}$.

Solution by the Rev. T. R. TERRY, M.A.; Professor ZERR; *and others.*



Let θ be the angle turned through at any time before the bead slides, M and m the masses of the sphere and bead, and a the radius of the sphere.

The equation of energy is

$$(2M+5m)a\theta'^2 = 10mg(1-\cos\theta)\ldots(1);$$

$$\therefore\quad (2M+5m)\,a\theta'' = 5mg\sin\theta\ldots(2).$$

If R be the pressure and F the friction on the bead,

$$ma\theta'' = mg\sin\theta - F\ \ldots\ldots(3),$$

$$ma\theta'^2 = mg\cos\theta - R\ \ldots\ldots(4).$$

Substituting, $\qquad (2M+5m)\,F = 2Mmg\sin\theta,$

$$(2M+5m)\,R = mg\left\{(2M+15m)\cos\theta - 10m\right\};$$

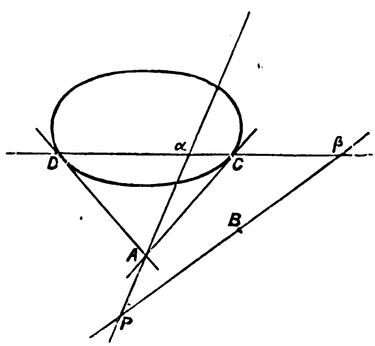
$$\therefore\quad F:R = 2M\sin\theta : (2M+15m)\cos\theta - 10m.$$

With the given numerical values, we have

$$\sin\theta = 2\cos\theta - 1;\qquad \therefore\quad (\tan\theta-2)^2 = 1+\tan^2\theta;\qquad \therefore\quad \tan\theta = \tfrac{3}{4}.$$

12283. (W. J. DOBBS, B.A.)—A and B are two fixed points. AP and BP are conjugate straight lines with respect to a fixed conic; find (1) the locus of P; and (2) examine the cases in which either or both of the points A and B are on the fixed conic; also when AB is a tangent to the conic.

Solution by Professors DROZ-FARNY, LAMPE, *and others.*



Lorsqu'un rayon Aα tourne autour d'un point fixe A son pôle β décrit une ponctuelle homographique au faisceau A sur la polaire CD du point A. Il en résulte que les faisceux Aα et Bβ sont homographiques et que le lieu du point P est une conique passant par A et B ainsi que par les points de contact des tangentes que l'on peut mener de A et de B à la conique proposée. Soit π le pôle de AB par rapport à la conique donnée, les droites πA et πB seront tangentes en A et B au lieu de P.

Si le point A est sur la conique, le lieu de P sera une conique tangente en A à la proposée ; si A et B sont tous deux sur la conique proposée, le lieu de P sera une conique doublement tangente à la proposée en A et B.

Si enfin la droite AB est tangente à la conique donnée, le lieu de P se dédouble en la droite AB et la droite qui joint les points de contact des deux secondes tangentes menées de A et B à la courbe donnée.

7624. (Professor SARKAR.)—If a train, consisting of p carriages, each of which will hold q men, contains $pq-m$ men, find the chance that another man B. getting in, and being equally likely to take any vacant place, will travel in the same carriage with a given passenger A.

9363, 12358. (Rev. T. C. SIMMONS, M.A.)—A railway carriage, consisting of three compartments, each of which will hold three persons, contains three passengers A., P., Q. A fourth passenger B., equally likely to take any vacant place, being now supposed to get in, show that his chance of travelling in the same compartment as A. is *not* $\frac{9}{32}$, and point out the fallacy in the solution given in Vol. XLVII., p. 74.

Solution by Rev. T. C. SIMMONS, M.A.

Quest. 7624 is of peculiar interest, considerable discussion having arisen years ago as to whether the answer was, or was not, independent of m. For my own part, I held it to be one of the simplest problems ever proposed, and that on first principles the answer was at once seen to be $(q-1)/(pq-1)$. Mr. BIDDLE and Mr. PUTNAM, on the other hand, maintained that I was entirely wrong, and that the problem was one of very great intricacy. Suppose, for instance, that $p = 60$, $q = 1000$, and $m > 1000$. According to these gentlemen, the answer will here necessitate the addition of 999 vulgar fractions, each containing some 4600 figures in the denominator, and with numerators ranging from some 4500 figures downwards; that is to say, in ordinary print, the *addition of* 999 *vulgar fractions, each more than* 30 *feet long!* (see Vols. XLV., p. 31; XLVI., p. 37; XLVII., p. 73). On the other hand, *without putting pen to paper, I declare the answer at once to be* $\frac{999}{59999}$. Now, which of us is right? The point at issue is too interesting and important to be left undecided, and, as nobody else has finally disposed of it, I propose to do so myself, and to prove, *beyond all possibility of a shadow of doubt, that the answer* $(q-1)/(pq-1)$ *is in all cases correct.*

It will greatly simplify matters, without affecting the generality of the argument, if we take a representative particular case. Imagine, then, a very common type of train, 10 carriages, each of 50 seats. When B. enters, let us suppose that 282 passengers, including A., have already seated themselves at random; what is the chance that B. and A. both find themselves in the same carriage?—I will give five independent solutions, each proving it to be $\frac{49}{499}$.

(1) If A. had been the first to enter the train, and B. second, B.'s chance of consorting with A. would have been $\frac{49}{499}$. But, the 283 pas-

sengers being all distributed at random, the chance of any two consorting together is clearly not affected by the order in which they enter the train. Therefore the chance of B. consorting with A. is always $\frac{49}{499}$. This is merely another wording of my original solution in Vol. XLV., wherein it is stated that, *A. being supposed already seated, the chance that any subsequent passenger will occupy any particular seat is clearly the same whether he enter the train first or last of the remaining* 282 *men.* Mr. BIDDLE objects to the italicised statement, which he calls "misleading to assert as having any bearing on the question." It holds true, he says, if there are 499 vacancies, or if there is only one vacancy, at B.'s disposal; but "in the intermediate cases the chance varies, being greater than at the extremes." Mr. PUTNAM, in different words, raises the same objection. I proceed, therefore, to another solution.

(2) It will, I suppose, be admitted that, of the 499 seats untenanted by A., *there is no reason why B. should occupy any one rather than any other.* Therefore his chance of occupying any particular one (for instance, the next seat to A. on the right) is $\frac{1}{499}$. Now, in 49 of these cases he will be A.'s consort, in the other cases he will not; therefore the chance required is $\frac{49}{499}$. The statement in italics is to me axiomatic; but experience has shown me that what is self-evident to one man is not self-evident to another. We will therefore continue.

(3) Suppose the seats all numbered, from 1 to 500. Now there is no reason why A. or B. should occupy any particular number more than any other. Therefore the chance that they occupy any particular pair of numbers is $\frac{1 \,.\, 2}{500 \,.\, 499}$. But of these pairs of numbers, $10 \,.\, \frac{50 \,.\, 49}{2}$ are favourable to their consorting, since there are 10 carriages, and $\frac{50 \,.\, 49}{2}$ pairs of numbers in each. Therefore chance required

$$= \frac{1 \,.\, 2}{500 \,.\, 499} \times \frac{500 \,.\, 49}{2} = \frac{49}{499}.$$

(4) Hitherto, I have taken no account of the other 281 passengers, because they have no influence on the result. If any one objects to this, and thinks they ought to be considered; be it so. The chance that, before B.'s entrance, the seat next A. on the right is occupied is $\frac{281}{499}$, the chance that it is unoccupied is $\frac{218}{499}$. In the first case, B. does not choose it; in the latter case his chance of choosing it among the 218 vacant seats is $\frac{1}{218}$. Therefore chance that B. sits next to the right of A. is $0 \,.\, \frac{281}{499} + \frac{1}{218} \,.\, \frac{218}{499}$ or $\frac{1}{499}$. The rest of the solution is the same as in (2), giving for result $\frac{49}{499}$.

(5) Mr. BIDDLE's and Mr. PUTNAM's difficulty arose, perhaps, from the difficulty of conceiving passengers in any railway train to seat themselves at random: it is so contrary to all experience. Brown's chance of travelling in the same carriage with his friend Jones (already seated) does, in actual life, depend very much on whether the train is full or empty; so that it seems almost paradoxical, even when the seats are chosen at random, to say that the result has no dependence on the number of the other passengers. This is, however, an unquestionable (I had almost said *undoubted*) fact, as may be seen by supposing a bag to contain balls

of p different colours, q balls of each colour, and the seats to be determined by drawing. First, $pq-m$ balls are drawn, including A.'s; B. then draws, and it will make no difference if we suppose the remaining balls to be drawn subsequently by $m-1$ fresh comers. We have now pq balls drawn successively by pq different people. All the people who draw the same colour seat themselves in the same carriage. *Now the chance that A. and B. draw balls of the same colour is most certainly* $(q-1)/(pq-1)$, *and has nothing to do with the order of drawing.* Therefore, &c.

(6) And now for Mr. BIDDLE's solution. The required chance is, he says (Vol. XLV., p. 31)

$$\frac{(pq-q)!\,(pq-m-1)!}{(pq-1)!\,(pq-m-q)!}\left\{\frac{q-1}{pq-q-m+1}+\frac{(q-1)(q-2)(m-1)}{(pq-q-m+1)(pq-q-m+2)}\right.$$
$$\left.+\frac{(q-1)(q-2)(q-3)(m-1)(m-2)}{(pq-q-m+1)(pq-q-m+2)(pq-q-m+3)}+\dots\right\}$$

to m terms; or (if $q-1$ be less than m) to $q-1$ terms, and is, I believe, quite correct; but what a terrible formula to apply when both m and q are at all large! He has, however, tried it for the comparatively simple case of $p=20$, $q=10$, $m=4$, and has made a numerical error. His 62,759,720 ought to be 60,759,720. If he subtracts this 2,000,000 from his final numerator, and then cancels by the G.C.M. 7645176, he will find the result to be $\frac{9}{199}$, an unexpected testimony to the correctness of my own solution on the same page. I have no doubt that the sum of the above series is, in fact, always $(q-1)/(pq-1)$; and it would be interesting to show this by treating it as a new question.

(7) A word or two will suffice for Quest. 9363. On p. 74 of Vol. XLVII., Mr. BIDDLE makes a series of most extraordinary blunders. After giving the above correct formula (a very difficult matter) for the general case, it is surprising that he should drift into a grievous fallacy in solving such a simple case as that of 3 compartments, each 3 seats, only one other passenger besides A. and B.; declaring the answer to be $\frac{13}{48}$, and my answer of $\frac{1}{4}$, given in Vol. XLVI., p. 38, to be wrong. Why, his own above-stated formula proves the result in all the three instances on that page to be $\frac{1}{4}$, as it ought to be!

I must apologise for treating at such length a question which some readers may deem to be very simple; but nothing is, to my mind, more unsatisfactory than these probability discussions which leave the reader finally uninformed. Mr. BIDDLE and Mr. PUTNAM will, I think, now at any rate, admit that *the chance of A. and B. travelling together is unaffected by the order in which the several passengers enter the train, and is the same as if there were no other passengers at all;* and so a most interesting problem is settled once and for ever.

[Mr. BIDDLE remarks that, as his initial formula (Vol. XLV., p. 30) is now admitted to be correct, though intricate and cumbrous, he readily confesses to the numerical error which Mr. SIMMONS has pointed out. It appears to have been an error of transcription, otherwise the subsequent terms in the numerator of P would have been vitiated, which is not the case. But for this error, the agreement of his formula with the much simpler one of Mr. SIMMONS would at once have been apparent, and all further mistakes would have been obviated. At the same time, he does not consider it so easy to see, that, although m must have integral

existence,—for, if it were zero, the probability of B. meeting A. would be 0 instead of $(q-1)/(pq-1)$,—yet its value, from 1 to $pq-1$, is a matter of indifference. We are under obligation to Mr. SIMMONS for showing clearly that such is the case.]

12323. (J. J. BARNIVILLE, B.A.)—Prove that

$$1-\frac{1}{3}-\frac{1}{4}+\frac{1}{6}+\frac{1}{7}-\frac{1}{9}-\ldots=\frac{\pi}{3\sqrt{3}},$$

$$\frac{1}{1.4.7}+\frac{1}{13.16.19}+\frac{1}{25.28.31}+\ldots=\frac{1}{36}\left(\log 2+\frac{\pi}{3\sqrt{3}}\right),$$

$$\frac{1}{1.2.3}+\frac{1}{9.10.11}+\frac{1}{17.18.19}+\ldots=\frac{1}{8}\left(\log 2+\frac{\pi}{2+\sqrt{8}}\right).$$

Solution by the PROPOSER, REV. T. ROACH, M.A., *and others.*

1. We have $$1+\frac{1}{2}-\frac{1}{4}-\frac{1}{5}+\frac{1}{7}+\ldots=\frac{2\pi}{3\sqrt{3}},$$

and $$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{5}+\frac{1}{7}-\ldots=\frac{2}{3}\log 2,$$

$\therefore$ $$1-\frac{1}{4}+\frac{1}{7}-\frac{1}{10}+\ldots=\frac{1}{3}\left(\frac{\pi}{\sqrt{3}}+\log 2\right),$$

and $$\frac{1}{2}-\frac{1}{5}+\frac{1}{8}-\frac{1}{11}+\ldots=\frac{1}{3}\left(\frac{\pi}{\sqrt{3}}-\log 2\right);$$

but $$\frac{1}{3}-\frac{1}{6}+\frac{1}{9}-\frac{1}{12}+\ldots=\frac{1}{3}\log 2,$$

$\therefore$ $$1-\frac{1}{3}-\frac{1}{4}+\frac{1}{6}+\frac{1}{7}-\ldots=\frac{\pi}{3\sqrt{3}},$$

and $$\frac{1}{2}+\frac{1}{3}-\frac{1}{5}-\frac{1}{6}+\frac{1}{8}+\ldots=\frac{\pi}{3\sqrt{3}}.$$

2. Here $$\frac{18}{1.4.7}+\frac{18}{7.10.13}+\frac{18}{13.16.19}+\ldots=\frac{2\pi}{3\sqrt{3}}+\frac{2}{3}\log 2-1,$$

and $$\frac{18}{1.4.7}-\frac{18}{7.10.13}+\frac{18}{13.16.19}-\ldots$$

$$=1+\frac{1}{7}-\frac{1}{7}-\frac{1}{13}+\ldots-\left(\frac{1}{2}-\frac{1}{5}+\frac{1}{8}-\ldots\right)$$

$$=1-\frac{\pi}{3\sqrt{3}}+\frac{1}{3}\log 2,$$

$\therefore$ $$\frac{1}{1.4.7}+\frac{1}{13.16.19}+\frac{1}{25.28.31}+\ldots=\frac{1}{36}\left(\log 2+\frac{\pi}{3\sqrt{3}}\right).$$

3. We have $$1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\ldots=\frac{\pi}{2\sqrt{2}},$$

$$\therefore \quad \frac{1}{1.2.3}-\frac{1}{5.6.7}+\frac{1}{9.10.11}-\ldots=\frac{\pi}{8(1+\sqrt{2})};$$

but $$\frac{1}{1.2.3}+\frac{1}{5.6.7}+\frac{1}{9.10.11}+\ldots=\frac{1}{4}\log 2,$$

$$\therefore \quad \frac{1}{1.2.3}+\frac{1}{9.10.11}+\frac{1}{17.18.19}+\ldots=\frac{1}{8}\log 2+\frac{\pi}{16(1+\sqrt{2})}.$$

$$\frac{1}{5.6.7}+\frac{1}{13.14.15}+\frac{1}{21.22.23}+\ldots=\frac{1}{8}\log 2-\frac{\pi}{16(1+\sqrt{2})}.$$

3029. (S. Watson.)—Find the locus of the intersection of normals at the extremities of focal chords in an ellipse.

Solution by R. Chartres.

By a well-known problem, Milne's *Companion*, p. 256, the locus of the intersections of normals at the ends of a focal chord of a conic is a conic of the same species, having its major axis coinciding with that of the original conic, and $= (1+e^2)\,ae$.

12360. (Artemas Martin, LL.D.)—From each of two equal coins a coin is cut at random. If one of these random coins be placed on the other at random, find the probability that the top coin will not fall off.

Solution by H. W. Curjel, B.A.

The chance that the coin cut off a coin of radius r will have a radius x is proportional to the area from which its centre must be taken, *i.e.*, to $\pi(r-x)^2$; therefore the chance that the top coin will not fall off

$$=\int_0^r\int_0^r \frac{x^2}{(x+y)^2}(r-x)^2(r-y)^2\,dx\,dy \Big/ \int_0^r (r-x)^2(r-y)^2\,dx\,dy$$

$$=9\int_0^1\int_0^1 \frac{x^2}{(x+y)^2}(1-x)^2(1-y)^2\,dx\,dy$$

$$=9\int_0^1\left[2x^4-3x^3+x-2(x^2-x^3-x^4+x^5)(\log\overline{1+x}-\log x)\right]dx$$

$$=\frac{73-96\log 2}{20}.$$

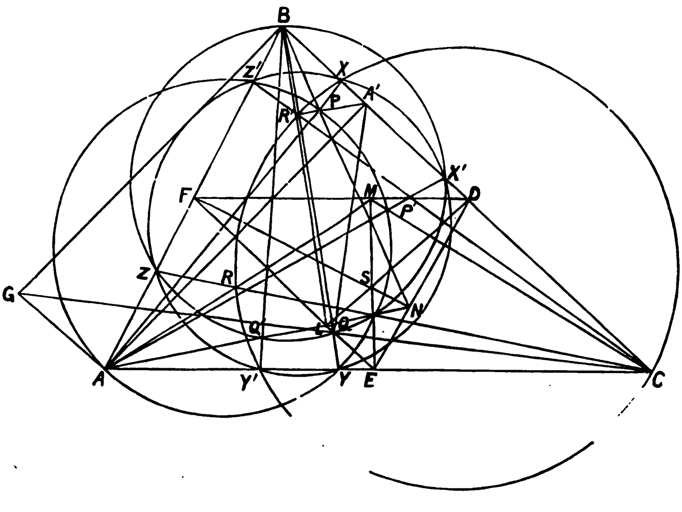
If all radii are equally likely for the coins, the chance

$$=\int_0^1\int_0^1 \frac{x^2}{(x+y)^2}\,dx\,dy = 1-\log 2.$$

12138. (H. W. Curjel, B.A.)—The cosine circle of the triangle ABC cuts the sides BC, CA, AB in XX′, YY′, ZZ′; XY′, YZ′, ZX′ being diameters; P, P′ are the intersections of AX, AX′ with the circle AYZ′; Q, Q′, R, R′ are similarly determined. Show that P′, Q′, R′ are the isogonal conjugates of R, P, Q with respect to the triangle ABC. Also, if CQ, BR′ cut in L, AR, CP′ in M, and BP, AQ′ in N, show that L, M, N are the feet of the perpendiculars from the circumcentre of ABC on the sides of the triangle formed by joining the middle points of the sides of ABC. [The point Q is the same as in Question 11593.]

Solution by the Proposer.

As in Solution to Question 11593 (Vol. lx., p. 115), ∠ BQA is a right angle and ∠ BQC = ∠ A + ∠ C; therefore circle BQC touches AB.



Similarly, ∠ CR′A is a right angle, and circle BR′C touches AC.

Hence ∠ QAC = complement of ∠ BYA and ∠ R′AB = the complement of ∠ CZ′A. But Y, Z′, B, C are concyclic; ∴ ∠ CZ′B = ∠ BYC; ∴ ∠ CZ′A = ∠ BYA; ∴ ∠ R′AB = ∠ QAC; ∴ ∠ R′AC = ∠ QAB.

Thus ∠ QCB = ∠ QBA = complement of ∠ QAB
= complement of ∠ R′AC = ∠ ACR′ = ∠ CBR′;

∴ Q, R′ are isogonally conjugate with respect to △ABC.

Similarly, R, P are isogonal conjugates of P′, Q′.

Again, since ∠ R′BC = ∠ QBA = ∠ QCB, ∴ BL = CL; ∴ SL is perpendicular to BC, S being the circumcentre.

If A′ is foot of perpendicular from A on BC,

$$\angle R'A'A = \angle R'CA = \angle R'BC = \angle R'BA';$$

∴ ∠BR′A′ is a right angle. Similarly, ∠CQA′ is a right angle. Produce CQ to meet BG, the perpendicular to BC through B, in G. Then circle BA′Q passes through G, but it also passes through A; ∴ ∠AGB is a right angle; ∴ AA′ = GB, but GC is bisected in L; ∴ L is in the line joining E, F the middle points of AC, AB, and is therefore the foot of the perpendicular from S on EF. Similarly, M, N are the feet of the perpendiculars from S on DF, DE where D is the middle point of BC.

11673. (H. J. Woodall, A.R.C.S.)—Place 4 sovereigns and 4 shillings in close alternate order in a line. Required in four moves, each of two contiguous pieces (without altering the relative positions of the two), to form a *continuous* line of 4 sovereigns followed by 4 shillings.

Solution by J. Gilbart Smyly, M.A.

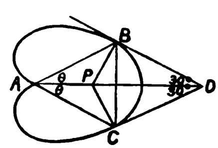
The Solution given by Mr. Davis on p. 45 of Vol. lix. is wrong, for, if the pieces be arranged as stated, A*a*B*b*C*c*D*d*, A*a*B ... *c*D*db*C, first move.

Then, in order to obtain the next order given, AB*ca*D*db*C, it is necessary to move not only the two pieces B*c* which are no longer contiguous, but also the single piece *a*. The following is the correct solution:—

A*a*B*b*C*c*D*d*,
A   *b*C*c*D*da*B   1st move,
AC*cb*   D*da*B   2nd ,,
AC*cbda*D   B   3rd ,,
*cbda*DACB   4th ,,

12320. (R. Chartres.)—If in any closed curve, symmetrical with regard to the initial line, an isosceles triangle be inscribed with its vertex A at the pole or cusp, find its position that will give the minimum Σ (FA) its maximum value, F being Fermat's point. Find the vertical angle of the triangle in the curve $r^n = a^n \cos n\theta$, and show that for the lemniscata, circle, and cardioid the angles are in arithmetical progression.

Solution by the Proposer; Professor Chakrivarti; *and others.*



Let ABC be an isosceles, and BCD an equilateral triangle; then, if P be Fermat's point, so that Σ (PA) is a minimum, then

$$\Sigma (PA) = AD,$$

and this is evidently a maximum when BD and CD are tangents, as can also be easily shown analytically.

In the curve $r^n = a^n . \cos n\theta$, $\tan \text{ABD} = \tan\phi = \dfrac{r\,d\theta}{dr} = -\cot n\theta$,

but $$\phi + \theta = 150^\circ, \quad \therefore \quad \theta = \frac{60^\circ}{n+1}.$$

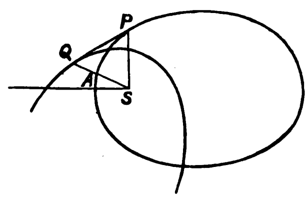
(1) If $n = 2$, we have for the lemniscata $\theta = 20^\circ$.
(2) If $n = 1$, ,, ,, circle $\theta = 30^\circ$.
(3) If $n = \frac{1}{2}$, ,, ,, cardioid $\theta = 40^\circ$.

12389. (J. W. Russell, M.A.)—Two equal conics are at first superposed. One of them is fixed and the other rotates about a common focus. Show that (1) the locus of the point of contact with the moving conic of a common tangent of the two conics is

$$lu = (1-e^2)(1-e\cos\theta)/(1-2e\cos\theta+e^2),$$

and (2) interpret the result when $e = 1$.

Solution by H. Fortey; Professor Mukhopadhyay; *and others.*



(1) Let S be the common focus of the conics, and P, Q the points of contact of a common tangent to the fixed and moving conics.

Let $l = \frac{1}{2}$ latus rectum,

$\angle \text{ASP} = \alpha$, $\text{SP} = r = u^{-1}$.

Then for P,

$$lu = 1 + e\cos\alpha,$$

and the equation to PQ is

$$lu' = \cos(\alpha - \theta) + e\cos\theta.$$

But from the symmetry it is clear that SQ = SP. Therefore for Q, $u' = u$ and $\cos(\alpha-\theta) + e\cos\theta = 1 + e\cos\alpha$. Solving for $\cos\theta$, we have

$$\cos \text{ASQ} = \cos\theta = \frac{2e + (1+e^2)\cos\alpha}{1+e^2+2e\cos\alpha},$$

whence $$\cos\alpha = \frac{(1+e^2)\cos\theta - 2e}{1+e^2-2e\cos\theta} = \frac{lu-1}{e}, \quad lu = \frac{(1-e^2)(1-e\cos\theta)}{1-2e\cos\theta+e^2}.$$

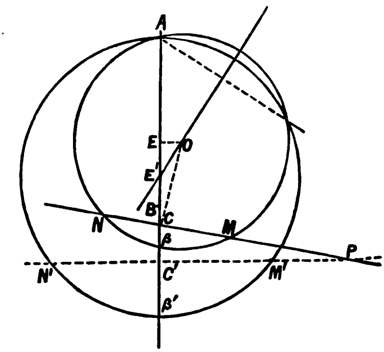
(2) When the curves are parabolas, putting $e = 1$ in the value of $\cos\theta$ above, we have $\cos\theta = 1$, or $\theta = 0$ for all values of α. Therefore Q is on the axis produced of the fixed parabola and varies from SA to infinity. And, putting $\theta = 0$ in the equation to the locus, we have

$$lu = \frac{(1-e^2)(1-e)}{(1-e)^2} = (1+e)\frac{(1-e)^2}{-e)^2} = 2\frac{0}{0},$$

when $e = 1$; therefore r is indeterminate, which is correct.

12342. (Professor Veyre.)—On donne une droite mobile autour d'un point P et deux points fixes A et B extérieurs. On trace les deux cercles tangents à MN en M et N et passant par A et B. Démontrer que la circonférence passant par M, N, et A (ou B) passe par un second point fixe.

Solution by Professor Droz-Farny; H. W. Curjel, M.A.; *and others.*



Soit PMCN une position de la droite mobile qui coupe la droite AB en C. Comme

$$(CM)^2 = (CN)^2 = CA \,.\, CB,$$

on a évidemment

$$CM = CN.$$

Soit β le symétrique de B par rapport à C; comme

$$CA \,.\, C\beta = CM \,.\, CN,$$

le quadrilatère AMβN sera inscriptible. E étant le milieu de la droite Aβ, les perpendiculaires EO sur AB et OC sur MN se coupent en O, centre de cette circonférence.

Considérons une seconde position PM'C'N' de la sécante perpendiculaire sur AB, et soit comme précédemment $BC' = C'\beta'$, et $AE' = E'\beta'$; E' sera le centre de la circonférence AM'N'β'.

Comme $$AE' = \tfrac{1}{2}AB + BC', \quad AE = \tfrac{1}{2}AB + BC,$$
$$EE' = CC' \quad \text{et} \quad EC = E'C' = \tfrac{1}{2}AB.$$

Dans les triangles semblables OEC et CC'P on a

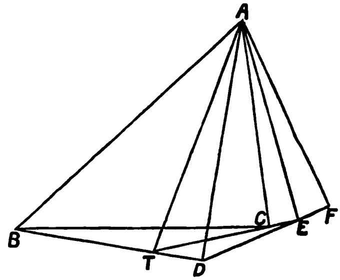
$$EO : EC = CC' : C'P, \quad \text{ou} \quad EO : EE' = EC : C'P,$$

ou $$EO : EE' = \tfrac{1}{2}AB : C'P = \text{constant};$$

l'angle OE'E étant constant, le lieu des centres des circonférences AMN est la droite E'O, et par conséquent toutes ces circonférences passent par un second point fixe symétrique de A par rapport à la droite E'O.

12383. (R. F. Davis, M.A.)—If upon the internal bisector of the angle A of a triangle ABC, a point T be taken such that $AT^2 = AB \,.\, AC$, prove that (1) the latus rectum ($4l$) of the parabola described, having A as focus and touching TB, TC at B, C respectively, is given by the equation $\{(s-b)(s-c)\}^{\frac{1}{2}} = \{s^{\frac{1}{2}} - (s-a)^{\frac{1}{2}}\}\, l^{\frac{1}{2}}$; and (2) the area of the parabolic sector ABC is $\frac{1}{3}l^{\frac{1}{2}}\{s^{\frac{3}{2}} - (s-a)^{\frac{3}{2}}\}$.

Solution by H. W. CURJEL, B.A.; Prof. MUKHOPADHYAY; *and others.*



Draw AD, AE perpendicular to BT, TC and AF perpendicular to DE. Then AF $= l$. Let

$$\angle \mathrm{ABT} = \phi, \quad \mathrm{BTA} = \psi.$$

Then $l/c = \sin^2\phi$, $l/b = \sin^2\psi$,

and $\tan\dfrac{\psi-\phi}{2} = \dfrac{c^{\frac12}-b^{\frac12}}{c^{\frac12}+b^{\frac12}}\cot\tfrac12\mathrm{A}$;

$$\therefore\ l/b - l/c = \sin(\psi-\phi)\sin(\psi+\phi)$$

$$= \sin^2\tfrac12\mathrm{A}\,\frac{c-b}{b+c-2(bc)^{\frac12}\cos\frac12\mathrm{A}}.$$

Hence we have $\qquad \dfrac{(s-c)(s-b)}{b+c-2\{s(s-a)\}^{\frac12}} = \dfrac{(s-b)(s-c)}{\{s^{\frac12}-(s-a)^{\frac12}\}^2}$;

therefore $\qquad l^{\frac12}\{s^{\frac12}-(s-a)^{\frac12}\} = \{(s-b)(s-c)\}^{\frac12}.$

Area of parabolic sector

$$= \Delta\mathrm{ABC} + \tfrac23\Delta\mathrm{BCT} = \mathrm{S} + \tfrac23\left\{\tfrac12(b+c)\sin\tfrac12\mathrm{A}\,(bc)^{\frac12} - \mathrm{S}\right\}$$
$$= \tfrac13\left\{\left[s(s-a)(s-b)(s-c)\right]^{\frac12} + (b+c)\left[(s-b)(s-c)\right]^{\frac12}\right\}$$
$$= \tfrac13 l^{\frac12}\left\{s+s^{\frac12}(s-a)^{\frac12}+s-a\right\}\left\{s^{\frac12}-(s-a)^{\frac12}\right\} = \tfrac13 l^{\frac12}\left\{s^{\frac32}-(s-a)^{\frac32}\right\}.$$

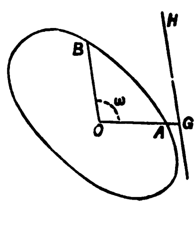
[*Otherwise:* $\mathrm{BD}^2 = c^2 + bc - 2c(bc)^{\frac12}\cos\tfrac12\mathrm{A} = c\{s^{\frac12}-(s-a)^{\frac12}\}^2$;

$$\mathrm{BD}^2.\,\mathrm{AT}^2 = c^2.\,bc\,.\sin^2\tfrac12\mathrm{A} = c^2(s-b)(s-c);$$

but from the property $\mathrm{SY}^2 = \mathrm{SA}.\mathrm{SP}$ in the parabola $\mathrm{AT}^2 = lc$; whence, &c.]

2805. (G. O'HANLON.)—In a given straight line, find the point at which a given ellipse subtends the greatest possible angle.

Solution by Profs. TANJÁNA, M.A., BHATTACHARYA, *and others.*



Refer the ellipse to conjugate axes OA, OB, the latter parallel to the given straight line GH.

Let $\mathrm{OA} = a$, $\mathrm{OB} = b$, $\mathrm{OG} = g$.

Any point in GH will have the coordinates g, γ. The two tangents to the ellipse from this point are

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{g^2}{a^2}+\frac{\gamma^2}{b^2}-1\right) = \left(\frac{gx}{a^2}+\frac{\gamma y}{b^2}-1\right)^2.$$

The coefficients of x^2, $2xy$, y^2 are

$$\frac{1}{a^2}\left(\frac{\gamma^2}{b^2}-1\right), \quad -\frac{g\gamma}{a^2b^2}, \quad \frac{1}{b^2}\left(\frac{g^2}{a^2}-1\right).$$

Hence, ϕ being the angle between the tangents,

$$\tan\phi = \frac{2\sin\omega\left\{g^2\gamma^2/a^4b^4 - 1/a^2b^2(\gamma^2/b^2-1)(g^2/a^2-1)\right\}^{\frac{1}{2}}}{1/a^2(\gamma^2/b^2-1)+1/b^2(g^2/a^2-1)+2g\gamma/a^2b^2\cos\omega}$$
$$= \frac{2\sin\omega\,(a^2\gamma^2+b^2g^2-a^2b^2)^{\frac{1}{2}}}{\gamma^2+2g\gamma\cos\omega+g^2-a^2-b^2}.$$

Equate $d\tan\phi/d\gamma$ to zero; we get

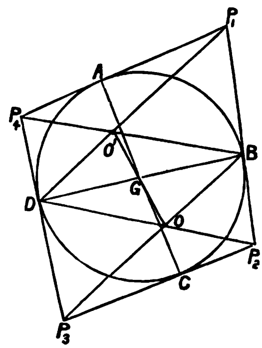
$$a^2\gamma^3+\gamma(a^4-a^2g^2-a^2b^2+2b^2g^2)+2b^2g(g^2-a^2)\cos\omega = 0.$$

This determines the required point.

It is obvious that the ordinates of two such points on one side of the axis of x are together equal to the ordinate of the remaining point. The two extreme positions determined will give maxima values of ϕ; the mean position will determine a minimum.

12325. (R. Knowles, B.A.)—The diagonals AC and BD of a quadrilateral inscribed in a conic meet in G; P_1, P_2, P_3, P_4 are the poles of the sides AB, CB, DC, AD respectively; P_2D and P_4B meet in O; P_1D and P_4B in O′; prove that O, O′, G are collinear.

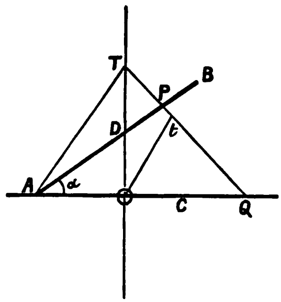
Solution by W. J. Dobbs, M.A.; Professor Droz-Farny; *and others.*



Project the conic into a circle having its centre at G. Then $P_1P_2P_3P_4$ projects into a rhombus touching the circle at A, B, C, D. The whole figure is thus symmetrical about G as a centre of symmetry. Also O and O′ are corresponding points; and therefore O, G, O′ are collinear.

7308. (Editor.)—A rectilinear beam APB, resting on the horizontal plane AQC at the point A, is supported in a given position by a prop of given length PQ; find (1) in what position PQ sustains the least pressure, and (2) what force is requisite to keep the beam in its position.

Solution by H. J. Woodall, A.R.C.S.; Prof. Cha rivarti; *and others.*



Denote AB, PQ, BAC, AP by $2b$, a, α, and x, respectively. Then

$$AP/\sin Q = AQ/\sin P = a/\sin\alpha$$
$$= \text{constant} = k, \text{ say}.$$

Let D be midpoint of AB; through D draw DO vertical to meet AC in O; produce QP to meet OD in T. Join AT. The forces acting on the beam are weight along TO, reaction of support along PQ; hence reaction at A must pass through T (since these three forces are in equilibrium). Draw Ot parallel to AT. Therefore TOt is the triangle of forces.

By ordinary work it may be found that

$$\text{reaction at P : weight} = t\text{T : TO} = bk/\left\{x\tan\alpha\,(k^2-x^2)^{\frac{1}{2}}+x^2\right\}.$$

And from the condition that this is a maximum or a minimum, we find

$$x = k\cos\tfrac{1}{2}\alpha \quad \text{or} \quad x = k\sin\tfrac{1}{2}\alpha.$$

On substituting these values of x in tT : TO we get

$$2b/\left\{k\sin\alpha\tan\alpha + k\,(1+\cos\alpha)\right\} \quad \text{and} \quad 2b/\left\{k\sin\alpha\tan\alpha + k\,(1-\cos\alpha)\right\}.$$

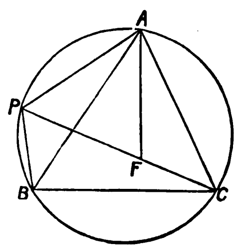
The former is a minimum, the latter a maximum.

The former reduces to $2b\cos\alpha/k\,(1+\cos\alpha) = b\cos\alpha/k\cos^2\tfrac{1}{2}\alpha$;

whence force along PQ $= \text{W}b\cos\alpha/k\cos^2\tfrac{1}{2}\alpha$, when $\text{AP} = x = k\cos\tfrac{1}{2}\alpha$.

12362. (J. A. Calderhead.)—If any point be taken in the circumference of a circle, and lines be drawn from it to the three angles of an inscribed equilateral triangle, prove that the middle line so drawn is equal to the sum of the other two.

Solution by T. Savage; W. J. Greenstreet, M.A.; *and others.*



Make angle CAF equal angle BAP Then

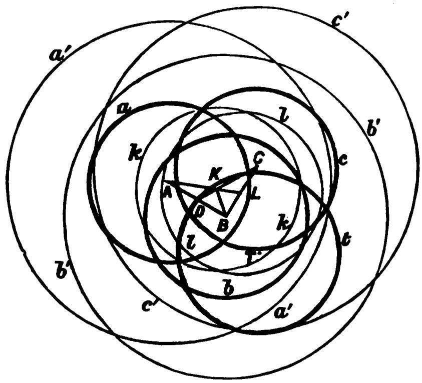
$$\text{PF} = \text{PA},$$

since triangle PAF is equilateral. Also (Euc. I. 26) $\quad$ CF = PB;

therefore $\quad$ PC = PA + PB.

5263 & 10905. (Artemas Martin, LL.D.)—**If four pennies be piled up at random on a horizontal plane, what is the probability that the pile will not fall down?**

Solution by D. Biddle.



Having recently received from the Proposer a copy of Mr. Heaton's solution of the above question, with a request for verification of the result (which itself alone occupies a page and a half quarto, in terms requiring double and treble integration), as published in the *Mathematical Magazine* for October, 1893, it seems worth while to consider the question *de novo*. The following note to the above-mentioned solution will prove of interest: "This famous 'Four-Pennies Problem' was proposed by the editor of the *Magazine* as the prize problem in *Our Schoolday Visitor Mathematical Almanac and Annual* for 1871, p. 47, but no solution was received. He also proposed it as a prize problem in the *Schoolday Visitor Magazine* for February, 1872, p. 54. No solution was received. The problem was also published in the *Educational Times* for April, 1877, as Quest. 5263, but no solution has yet appeared in that excellent journal or its *Reprint*. So far as known, Mr. Heaton is the only one who has effected a solution of this difficult problem."

In order to follow so far as possible Mr. Heaton's lettering, let t, T be respectively the penny next the table and its centre, whilst a, A, b, B, c, C are the second, third, and fourth pennies, and their centres. It is clear that C must lie over b, and, if L be the centre of gravity of b and c, midway between B and C, then L must lie over a; moreover, K, being the centre of gravity of a, b, c, a third of the distance from L to A, must lie over t. Now, D being the centre of gravity of a and b, midway between A and B, Mr. Heaton seems to make the problem more difficult than need be by saying, "The pile will stand if the pennies are so placed that A, D and K lie over the first penny (t), L, over the second (a), and C, over the third (b)." There is no need for A and D as well as K to lie over t. Theoretically, if the above conditions as to the three upper pennies taken by themselves be fulfilled, a pin's point applied underneath K is capable of supporting all three. For that K is not outside a is clear from the fact of its being in the line AL, both extremities of which are contained by a. Much more will any part of the upper surface of t, resting beneath K, support the three upper pennies, if they in turn properly support each other.

It is immaterial whether the arrangement of the pile take place from below upwards, or from above downwards. But there is an advantage in considering it as formed in the latter way. Let us therefore regard C as fixed, whilst the positions of B, A, T are variable. With centre C and radius equal twice that of a penny, describe a circle (c'). Then if B be anywhere within this circle, b will underlie c; but only when B is actually underneath c, is the latter supported by b, and, as c is one-fourth of the newly described circle concentric with it (c'), the probability that b supports c is $\frac{1}{4}$. So far the course is perfectly plain.

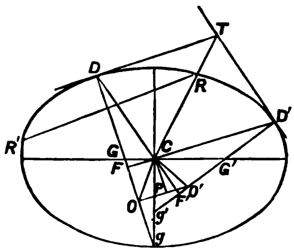
Now, in order that a may underlie b, it is similarly requisite that A should lie within a circle concentric with b, and of the same size as that above described (b'); but in order that a may support L, A must lie within a penny's radius of L. And if a circle (l) of such radius be drawn from L as centre, it will lie wholly within the larger circle (b') concentric with b, because B and L are less than half a penny's radius apart. Consequently, the subsidiary chance here is again $\frac{1}{4}$. In the same way, a circle (k) of a penny's radius drawn around K as centre, and indicating where T must lie in order that t may properly support the three superposed pennies, can easily be seen to lie wholly within the larger circle (a') concentric with a, that indicates the full tether of t (or rather of its centre T); for $AK = \frac{2}{3}AL$, and $AL <$ a penny's radius. Therefore the chance here again is $\frac{1}{4}$. Hence $P_4 = \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{64}$; and similarly, $P_n = (\frac{1}{4})^{n-1}$.

It will be observed that the special range is at first concentric with the total range, but becomes more and more eccentric, owing to the increasing distance of the centre of gravity of succeeding groups from the centre of the lowest in the group, as expressed by the fraction ($\frac{0}{1}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, &c.), until at last the two circles touch. But the smaller is always wholly contained by the larger. Therefore the chance every time is $\frac{1}{4}$.

[This solution, though made quite independently, is in substantial agreement with that given by Mr. Curjel on p. 114 of Vol. lxi., which solution was then unpublished. Mr. Heaton's solution seems to proceed upon the assumption that each penny must stand securely on that below it before another is placed above, but the unqualified term "random," as used in the question, does not sanction this limitation.]

12324. (F. S. Macaulay, M.A.)—Prove the following construction for drawing the four normals from any point to an ellipse whose periphery is given. Let the principal axes of the ellipse divide the plane into four quadrants, and let O be the given point; in the next quadrant to that in which O lies (in a definite direction of rotation) take a point O′ whose ordinate and abscissa bear to the abscissa and ordinate of O the ratios CA : CB and CB : CA respectively; bisect OO′ in P, and on CP take a point Q such that $CQ : CP = 1 : \sqrt{2}$; with Q as centre, describe a circle cutting the circle through the ends of the equi-conjugates along a diameter; let this circle cut the ellipse in points R; draw CR′ conjugate to CR, and in the next quadrant. Then the perpendiculars from O to the four chords RR′ are normal to the ellipse.

Solution by the PROPOSER.



Suppose OD a normal from O to the ellipse, let CD′ be the semidiameter conjugate to CD, so that CD is in the quadrant next to CD′ in the positive direction of rotation, let the normal at D cut the axes in G, g, and CD′ in F, and let G′, g', F′ be the similar points on the normal at D′. Take a point O′ in g'G′, such that $g'\mathrm{O}' : \mathrm{O}'\mathrm{G}' = \mathrm{GO} : \mathrm{O}g$. Then, if ON, O$n$ are the perpendiculars from O to the axes, and O′N′, O′n' the same for O′, we have

$$\mathrm{ON} : \mathrm{C}g = \mathrm{GO} : \mathrm{G}g = g'\mathrm{O}' : g'\mathrm{G}' = \mathrm{O}'n' : \mathrm{CG}'.$$

Hence $$\mathrm{O}'\mathrm{N}' : \mathrm{ON} = \mathrm{CG}' : \mathrm{C}g = b : a;$$
$$\mathrm{O}'\mathrm{N}' : \mathrm{O}n = \mathrm{C}g' : \mathrm{CG} = a : b;$$

hence O′ is the point mentioned in the enunciation. Let the tangents at D, D′ meet in T, let CT cut the curve in R, bisect OO′ in P, and let Q (not drawn in figure) be taken on CP, so that

$$\mathrm{CQ} : \mathrm{CP} = 1 : 2^{\frac{1}{2}};$$

then $$\mathrm{GO} : g'\mathrm{O}' = \mathrm{G}g : g'\mathrm{G}' = \mathrm{DG} : \mathrm{D}'\mathrm{G}' = \mathrm{D}'\mathrm{F}' : \mathrm{DF};$$

therefore $$\mathrm{DF} \,.\, \mathrm{GO} = \mathrm{D}'\mathrm{F}' \,.\, g'\mathrm{O}',$$

or $$\mathrm{DF} \,.\, \mathrm{DO} - \mathrm{DF} \,.\, \mathrm{DG} = \mathrm{D}'\mathrm{F}' \,.\, \mathrm{D}'g' - \mathrm{D}'\mathrm{F}' \,.\, \mathrm{D}'\mathrm{O}';$$

therefore $$\mathrm{DF} \,.\, \mathrm{DO} + \mathrm{D}'\mathrm{F}' \,.\, \mathrm{D}'\mathrm{O}' = a^2 + b^2;$$

but $$2\mathrm{DF} \,.\, \mathrm{DO} = \mathrm{CD}^2 + \mathrm{DO}^2 - \mathrm{CO}^2, \quad 2\mathrm{D}'\mathrm{F}' \,.\, \mathrm{D}'\mathrm{O}' = \mathrm{CD}'^2 + \mathrm{D}'\mathrm{O}'^2 - \mathrm{CO}'^2;$$

therefore $$\mathrm{CD}^2 + \mathrm{CD}'^2 + \mathrm{DO}^2 + \mathrm{D}'\mathrm{O}'^2 - \mathrm{CO}^2 - \mathrm{CO}'^2 = 2a^2 + 2b^2;$$

but $$\mathrm{CD}^2 + \mathrm{D}'\mathrm{O}'^2 = \mathrm{TD}'^2 + \mathrm{D}'\mathrm{O}'^2 = \mathrm{TO}'^2, \quad \mathrm{CD}'^2 + \mathrm{DO}^2 = \mathrm{TO}^2;$$

therefore $$\mathrm{TO}^2 + \mathrm{TO}'^2 - \mathrm{CO}^2 - \mathrm{CO}'^2 = 2a^2 + 2b^2,$$
$$\mathrm{TP}^2 - \mathrm{CP}^2 = a^2 + b^2;$$

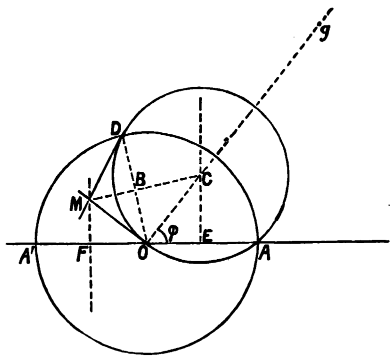
also $$\mathrm{CR} : \mathrm{CT} = \mathrm{CQ} : \mathrm{CP} = 1 : 2^{\frac{1}{2}};$$

therefore $$\mathrm{QR}^2 - \mathrm{QC}^2 = \tfrac{1}{2}(\mathrm{PT}^2 - \mathrm{PC}^2) = \tfrac{1}{2}(a^2 + b^2).$$

Hence the circle with centre Q which cuts the circle through the ends of the equi-conjugates along a diameter passes through R. Also, if CR′ be the semi-diameter conjugate to CR and in the next quadrant, it is evident that RR′ is parallel to TD, and hence OD is perpendicular to RR′, *i.e.*, the perpendiculars from O to the chords RR′ are normal to the ellipse.

12306. (Professor MANDART.)—Etant donnés un cercle O et un point A sur la circonférence, on décrit un cercle C par les points A, O et coupant le cercle O en D. Trouver (1) le lieu des points de rencontre M des tangentes menées en D et en O au cercle C; (2) le lieu des points de rencontre des tangentes communes aux deux cercles; et (3) l'enveloppe de la droite MC.

Solution by Professor Droz-Farny; W. J. Dobbs, M.A.; *an hers.*



(1) De M et C abaissons des perpendiculaires sur le diamètre OA; soient F et E leurs pieds respectifs et B le point d'intersection de MC avec OD.

CO étant la bissectrice de l'angle DOA, MO sera la bissectrice de l'angle extérieur DOA; donc les triangles rectangles MFO et MBO sont égaux; il en résulte

$$FO = OB = \tfrac{1}{2}OA = \tfrac{1}{2}R.$$

Le lieu de M est donc la perpendiculaire au diamètre AA' à une distance $OF = \tfrac{1}{2}R$.

(2) Soit sur OC, G le point d'intersection des tangentes communes aux circonférences O et C. Représentons par ρ et ϕ le rayon OG et l'angle COA.

On aura $$GC : GO = CO : 2OE = 1 : 2\cos\phi,$$

ou $$\rho - \tfrac{1}{2}R\sec\phi : \rho = 1 : 2\cos\phi, \quad \rho = 2R/(2\cos\phi - 1).$$

Le lieu de G est une hyperbole ayant O pour un foyer. Ses sommets sont situés sur le rayon OA à partir de O dans la direction OA à des distances $\tfrac{2}{3}R$ et 2R. Les demi-axes sont $\tfrac{2}{3}R$ et $\tfrac{2}{3}R\sqrt{3}$ et l'angle des asymptotes = 120°.

(3) Comme OF = OB = OE, l'enveloppe de CM est la circonférence de centre O et de rayon $\tfrac{1}{2}R$.

11975. (Editor.)—Find two numbers such that both their sum and difference shall be a square; also the sum of their squares shall be a cube, and the sum of their cubes a square.

Solution by H. W. Curjel, B.A.; R. Chartres; *and others.*

If x and y are the numbers, the condition that $x^3+y^3 = \square$ reduces to $x^2+y^2-xy = \square$, since $x+y = \square$; hence the conditions may be written

$$x+y = L^2, \quad y-x = M^2, \quad x^2+y^2 = N^3, \quad x^2+y^2-xy = P^2 \quad \ldots(1, 2, 3, 4).$$

(4) is satisfied by $\quad x = z(2mn-n^2), \quad y = z(m^2-n^2) \quad \ldots\ldots\ldots\ldots(5),$

where m, n, z are any integers, $m > n$, and P will be equal to $z(m^2-mn+n^2)$. (1) and (2) will be satisfied if z is a square number, and $m = 9$, $n = 4$. Then $x = z(56)$, $y = 65z$. Again, $56^2+65^2 = 7361$.

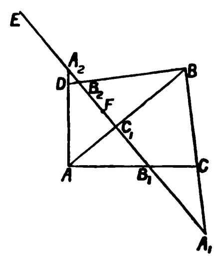
Hence the four equations (1), (2), (3), (4), are satisfied by

$$x = (7361)^4 \times 56, \quad y = (7361)^4 65,$$

also by xk^6, yk^6. Other values of x and y may be similarly deduced from the other values of m and n which satisfy (1) and (2) and (5).

12372. (Professor Neuberg.)—On considère toutes les coniques circonscrites à un triangle donné ABC divisant harmoniquement un segment donné EF. Ces courbes ont un quatrième point commun D, dont on demande une construction. Lorsque la droite EF et le point E sont fixes, mais que F se déplace, quel est le lieu décrit par D_1?

Solution by H. W. Curjel, B.A.; Professor Droz-Farny; *and others.*



Let EF cut the sides of the triangle ABC in A_1, B_1, C_1.

Find the harmonic conjugates A_2, B_2 of A_1, B_1 with respect to EF.

Let BB_2, AA_2 meet in D.

Since E, F are conjugate with respect to two conics (namely the line-pairs BC, AD; AC, BD) through A, B, C, D, they are conjugate with respect to all conics through ABCD. Hence all conics through ABC with respect to which EF are conjugate pass through D. For let U be any conic through ABC cutting EF harmonically in M, N; then the conic through ABCDN passes through M also, and therefore coincides with U.

Again, the conics through ABC touching EF, one at E and the other at F, divide EF harmonically, and therefore intersect at D. Therefore locus of D, when F moves along EF, is the conic through ABC touching EF at E.

11683. (Rev. Dr. Bruce.) — Show (1) how to place eight men on a draught-board so that no two of them shall be in line with one another, horizontally, perpendicularly, or diagonally; and find (2) in how many ways this can be done.

Note by R. Chartres.

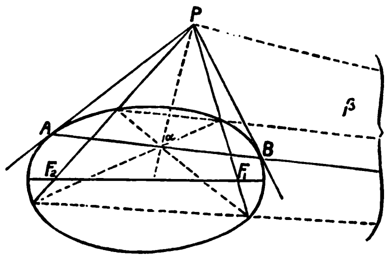
Dr. Bruce, in his solution to this Question (Vol. lix., p. 32), asks why in all cases the sum of the numbers just comes to 260. Now we have

$$1 + 2 + 3 \ldots + 64 = 260 \times 8;$$

hence the sum of any eight numbers, one from each row, and symmetrical in pairs as regards the columns, must give 260; and Dr. Bruce's numbers are all symmetrical in this way.

11932. (Professor Ignacio Beyens.)—On donne une ellipse, un point P qu'on joint aux foyers. Démontrer que les centres des sécantes communes au système des deux droites ainsi obtenues et à l'ellipse sont sur l'hyperbole d'Apollonius du point P.

Solution by Professors Droz-Farny, Zerr, *and others.*



Supposons le point P extérieur, et soient A et B les points de contact des tangentes menées de P à l'ellipse, et α et β les centres des sécantes communes au système des deux droites PF_1, PF_2 et à l'ellipse. $P\alpha$ est évidemment la polaire de β par rapport à l'éllipse; donc les droites $P\alpha$ et $P\beta$ sont séparées harmoniquement par les tangentes PA et PB. Il en résulte que ces deux droites $P\alpha$ et $P\beta$ sont les éléments doubles de l'involution des couples de tangentes que l'on peut mener de P aux coniques homofocales à l'ellipse proposée. Elles sont donc les tangentes aux deux coniques du système passant par P et se coupent par conséquent à angle droit. Les points α et β, étant tels que la perpendiculaire abaissée de l'un d'entre eux sur sa polaire passe par le point P, appartiennent à l'hyperbole d'Apollonius de ce point.

11789. (Professor Baschwitz.)—On a, identiquement,

$$\frac{1+x}{1-x}x+\frac{1+x^2}{1-x^2}x^4+\dots+\frac{1+x^n}{1-x^n}x^{(n^2)}$$

$$=\frac{x}{1-x}(1+x^n)+\frac{x^2}{1-x^2}(1+x^{2n})+\dots+\frac{x^n}{1-x^n}(1+x^{n^2}).$$

Si l'on suppose x compris entre 0 et $+1$, et que l'on fasse croître n indéfiniment, cette identité devient celle de Clausen.

Solution by H. J. Woodall, A.R.C.S.

Subtracting terms with similar denominators and expanding each such result, we get series such as the following

$$\frac{1+x}{1-x}x-\frac{x}{1-x}(1+x^n)=x^2+x^3+x^4+\dots+x^n,$$

$$\frac{1+x^2}{1-x^2}x^4-\frac{x^2}{1-x^2}(1+x^{2n})=-x^2+x^6+\dots+x^{2n},$$

$$\frac{1+x^r}{1-x^r}x^{r^2}-\frac{x^r}{1-x^r}(1+x^{rn})=-x^r-x^{2r}-x^{3r}\ldots-x^{r(r-1)}+x^{r(r+1)}+\ldots+x^{rn},\text{\&c.}$$

The resulting series in the rth line consists of Σx^{kr}, the sign being $\mp$ according as $k \lessgtr r$, the term x^{r^2} being absent. It is thus seen that each power of x, excluding such as x^{r^2}, occurs once each, with opposite signs. Hence the sum of the left-hand side equals the sum of the right-hand side, identically. The second part of theorem follows immediately.

8179. (Editor.)—Given two conics U, V; a chord PP′ of U is taken such that P, P′ are conjugate with respect to V; and a chord QQ′ of V such that Q, Q′ are conjugate with respect to U: prove that the envelope of PP′ coincides with that of QQ′, being the conic which touches the eight tangents drawn to U, V at their common points.

11008. (Professor Wolstenholme.)—Given two conics U, V, a point O is taken such that the two tangents drawn from O to U are conjugate with respect to V; prove that the tangents drawn from O to V will be conjugate with respect to U, and that the locus of O is the conic passing through the eight points of contact of the four common tangents to U, V.

Solution by H. W. Curjel, B.A.

(8179.) Let PP′ cut V in QQ′. Then, since P, P′ are harmonic conjugates with respect to V, *i.e.*, with respect to Q, Q′, therefore Q, Q′ are harmonic conjugates with respect to PP′; therefore with respect to U. Similarly, if Q, Q′ are conjugate with respect to U, P, P′ are conjugate with respect to V. Hence the two envelopes coincide. Projecting two of the points of intersection of U, V into the circles; U, V become circles and it is evident that the envelope of PP′ is the conic with its foci at the centres of the circles, and touching their tangents at their points of intersection. Hence in the original figure the envelope of PP′ is a conic touching the eight tangents drawn to U and V at their common points.

(11008.) This follows immediately from Quest. 8179 by reciprocation.

8834. (Professor Ignacio Beyens.)—Construct a triangle, knowing the length of a side and (1) an adjacent, (2) an opposite angle, the known side lying on a given right line and the other two sides passing through two given points.

Solution by H. J. Woodall, A.R.C.S.

1. Let Δ be the given line, A, B the points. From A draw AC making

with Δ the given angle α. Cut off CD the given length, join DB, and produce to cut CA produced.

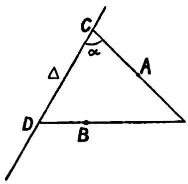


Fig. 1.

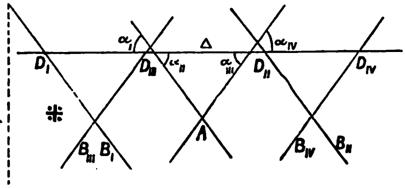


Fig. 2.

There may be more than one solution, according to which way the angle is laid down, and which way the length is laid off, *i.e.* four "possible" solutions (disregarding the direction of the angle α).

B on the same side of Δ as A.

Then, if B be on * side of	B_ID_I,	1 is possible;	triangle is	above Δ.
,, ,, ,,	$B_{II}D_{II}$,	2	,,	,,	below Δ.
,, opp. to * ,,	$B_{III}D_{III}$,	3	,,	,,	,,
,, ,, ,,	$B_{IV}D_{IV}$,	4	,,	,,	above Δ.

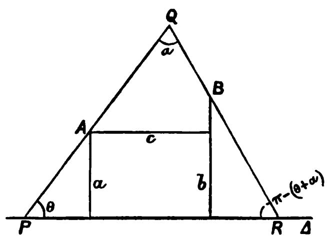
[1, 2, 3, or 4 refers to the triangle having its angle α in the position α_1, α_2, α_3, or α_4, respectively.]

B on the other side of Δ from A; the cases are reversed, the triangles being in the same position.

2. From AB draw perpendiculars to Δ; let their lengths be a, b, and the intercepted distance c. Through A draw PAQ, making an angle θ (away from B) with Δ; through B draw QBR, making an angle $Q = \alpha$ (given). Then

$$PR = a\cot\theta + c - b\cot(\theta+\alpha);$$

putting this equal to k (given), we get a quadratic in $\cot\alpha$, the two solutions of which will furnish solutions of the problem. And thus also the shortest length intercepted on Δ by PR.



12071. (J. Griffiths, M.A.)—Through each angular point of any triangle circumscribing a parabola a line is drawn parallel to the opposite side; prove that the new triangle formed by these three lines is self-conjugate with respect to the parabola. Hence, show that the nine-point circle of any triangle self-conjugate with respect to a parabola passes through the focus, and that the centre of its circumscribing circle lies on the directrix.

Solution by R. Knowles, B.A.; Professor Ignacio Beyens; *and others.*

Let x_1y_1, x_2y_2, x_3y_3 be the coordinates of the points of contact; the equations to the three sides of the triangle parallel to those of the original triangle are $\quad y-(y_1+y_2)/2 = 2a\,(x-y_1y_2/4a)/y_3,$

$$y-(y_2+y_3)/2 = 2a\,(x-y_2y_3/4a)/y_1, \quad y-(y_1+y_3)/2 = 2a\,(x-y_1y_3/4a)\,y_2\,;$$

and the coordinates of their points of intersection are

$$(y_1y_2+y_2y_3-y_1y_3)/4a,\, y_2\,;\; (y_1y_3+y_1y_2-y_2y_3)/4a, y_1\,;\; (y_1y_3+y_2y_3-y_1y_2)/4a,\, y_3.$$

It is easily seen that the polar of each of these points with respect to the parabola is its opposite side; therefore the triangle is self-conjugate with respect to the parabola. The mid-points of the sides are $y_1y_2/4a$, $(y_1+y_2)\,2$; $y_2y_3/4a$, $(y_2+y_3)^2/2$; $y_1y_3/4a$, $(y_1+y_3)/2$; therefore the nine-point circle of this triangle is the circumcircle of the original triangle, and it therefore passes through the focus. The equations to perpendiculars through these mid-points to the sides are

$$y-(y_1+y_2)/2 = -y_3\,(x-y_1y_2/4a)/2a, \quad y-(y_2+y_3)/2 = -y_1(x-y_2y_3/4a)/2a,$$

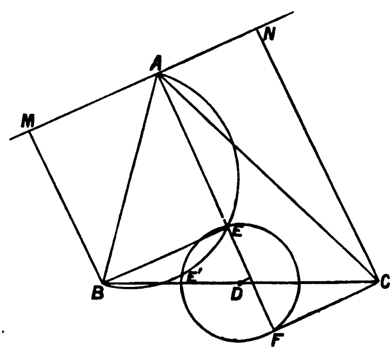
and these intersect in

$$x = -a,\; y = (y_1+y_2+y_3)/2 + y_1y_2y_3/8a^2\,;$$

therefore the centre of the circumcircle is in the directrix.

12403. (I. Arnold.)—Through the vertex of a triangle draw a right line, so that the rectangle under the perpendiculars upon it from the ends of the base shall be equal to a given square or rectangle, and show when the problem is impossible.

Solution by R. C. Abbott, B.A.; Professor Morel; *and others.*



To draw MAN, so that BM . CN may be equal to a given square. With centre D, the middle point of BC, describe a circle such that the tangent to it from A is equal to the side of the given square. Let the circle on AB as diameter cut this circle in E and E′. Draw AEF, and join BE, CF. Draw MAN perpendicular to AE. Then

BM . CN = AE . AF

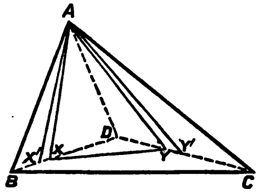
= given square.

The limiting case occurs when E and E′ coincide, giving one solution. If the circle EE′F does not meet AEB, there is no solution.

2614. (Colonel A. R. Clarke, C.B., F.R.S.)—Three points are taken at random, one on each of three faces of a tetrahedron; show that the chance that the plane passing through them cuts the fourth face is $\frac{3}{4}$.

8301. (Rev. T. C. Simmons, M.A.)—On each of three assigned faces of a tetrahedron, a random point is taken. Show that the plane thus determined has an even chance of cutting any given edge of the fourth face.

Solution by Rev. T. C. Simmons, M.A.



(8301.) Let the three assigned faces be those meeting at D in the tetrahedon ABCD. In BD take any point X, and in CD any point Y.

Let $DX = x$, $XX' = dx$;
$DY = y$, $YY' = dy$;
$DA = \alpha$, $DB = \beta$, $DC = \gamma$.

Then the chance that one point falls in $AXX' = dx/\beta$, and another point in $AYY' = dy/\gamma$; in which case, in order that the plane may cut the two lines AB, AC, it is necessary that the third point must lie within the triangle DXY, the chance of which is $xy/\beta\gamma$. Hence the whole chance of the plane meeting AB and AC

$$= \int_0^\beta \int_0^\gamma xy\,dx\,dy \Big/ \beta^2\gamma^2 = \tfrac{1}{4}.$$

Similarly, the chance of its meeting AB and $BC = \frac{1}{4}$; therefore, the two events being mutually exclusive, the whole chance of its meeting $AB = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$; and similarly for the chance of its meeting BC or AC.

(2614.) The chance of the plane cutting AB and AC, or BA and BC, or CA and CB, is in each case $\frac{1}{4}$; the three events are mutually exclusive; therefore the whole chance of its cutting the plane $ABC = \frac{3}{4}$.

[A very instructive solution of Quest. 2614, though much longer than this, has been given by Colonel Clarke himself, on p. 54 of Vol. x.]

12418. (Professor Draughton.)—Find the volume generated by revolving a circular segment, whose base is a given chord, about any diameter as an axis.

Solution by D. Biddle; Professor Mukhopadhyay; *and others.*

Let r be the radius of circle, α the angle formed by chord with radius at its extremity, and β the angle of inclination of chord to axis of revolution. Then the outer curved surface generated, and the portion of sphere enclosed, are

$$2\pi r^2 \left\{\cos(\alpha-\beta) + \cos(\alpha+\beta)\right\}, \quad \tfrac{2}{3}\pi r^3 \left\{\cos(\alpha-\beta) + \cos(\alpha+\beta)\right\}$$
$$+ \tfrac{1}{3}\pi r^3 \left\{\sin^2(\alpha-\beta)\cos(\alpha-\beta) + \sin^2(\alpha+\beta)\cos(\alpha+\beta)\right\},$$

From this must be deducted the frustum of cone generated by the chord and perpendiculars from its extremities upon the axis, namely,

$$\tfrac{1}{3}\pi r^3 \{\sin^2(\alpha+\beta)\left[\sin\alpha \operatorname{cosec}\beta + \cos(\alpha+\beta)\right] - \sin^2(\alpha-\beta)\left[\sin\alpha \operatorname{cosec}\beta - \cos(\alpha-\beta)\right]\},$$

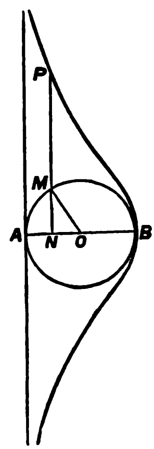
whence we obtain the result $= \tfrac{4}{3}\pi r^3 \cos^3\alpha \cos\beta = \pi/6 \cos\beta \times$ cube of chord.

12371. (Professor LAMPE, LL.D.)—Prove that the radius of curvature of the *Versiera*, $xy^2 + a^2x = a^3$, is $R = (a^4 + 4ax^3 - 4x^4)^{\frac{3}{2}} / \left[2x^2(3a-4x)a^2\right]$. The analytical method for minima leads to the equation

$$8x^5 - 12ax^4 + 5a^2x^3 + 2a^4x - a^5 = 0,$$

whence $x = 0{,}44516a$, $R = 2{,}7057a$. How is the fact to be explained that the evident minimum $R = \frac{1}{2}a$ for $x = a$ does not follow from this equation?

Solution by H. FORTEY; H. W. CURJEL, B.A.; *and others.*



In the diagram let AMB be the generating circle, and let $AB = a$, $AN = x$, and $PN = y$. Then, if $PN : AB = MN : AN$, the locus of P is the *Versiera*; and the equation to the curve, the value of R, and the equation resulting from $dR/dx = 0$ are all given in the question.

The reason that all the maxima and minima values of R do not result from $dR/dx = 0$ is that x is not an absolutely independent variable, being restricted by the limits 0 and a.

Let O be the centre of the circle; join OM, and let $\angle AOM = \theta$. Then θ varies without restriction, and

$$x = AN = a\sin^2\tfrac{1}{2}\theta.$$

Taking θ for independent variable, we have for maxima and minima of R,

$$\frac{dR}{d\theta} = \frac{dR}{dx}\cdot\frac{dx}{d\theta} = 0.$$

Now $dx/d\theta = 0$ gives $\sin\frac{1}{2}\theta \cos\frac{1}{2}\theta = 0$;

$$\sin\tfrac{1}{2}\theta = 0,\ \cos\tfrac{1}{2}\theta = 0,\ \text{give } x = 0,\ R = \infty;$$

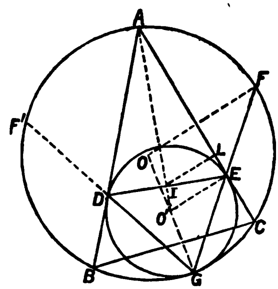
$$x = a,\ R = \tfrac{1}{2}a;$$

and these values are not obtainable from $dR/dx = 0$. For similar questions, see *Camb. Math. Journal*, Vol. III., p. 237; or TODHUNTER'S *Diff. Calc.*

12424. (EDITOR.)—Draw (1) four circles, each of which shall touch the circumcircle of a triangle ABC and the sides AB, AC; prove that (2) the radii of these circles are $r\sec^2\frac{1}{2}A$, $r_a\sec^2\frac{1}{2}A$, $r_b\operatorname{cosec}^2\frac{1}{2}B$,

r_c cosec$^2 \frac{1}{2}$C ; and (3) the poles of A with respect to these four circles pass through the in- and ex-centres of the triangle.

Solution by Professors Droz-Farny, Mukhopadhyay, *and others.*



Soit O′ le centre du cercle tangent en D et E aux côtés AB, AC et en G intérieurement au cercle circonscrit. Soit F le point milieu de l'arc AC ; comme OF et O′E sont parallèles, la droite FE passe par le centre de similitude directe G des deux circonférences O et O′. La circonférence O′ pouvant être considérée comme l'inverse de AC, on a $(FA)^2 = FE . FG$. De même F′ étant le milieu de l'arc AB se trouve sur GD et on a $(F'A)^2 = F'D . F'G$. La droite FF′ sera donc l'axe radical de la circonférence O′ et du cercle point A. Elle divise donc les tangentes AD et AE en parties égales et est perpendiculaire sur la bissectrice O′A de l'angle A. Il en résulte que les circonférences décrites de F et F′ comme centres avec respectivement FA et F′A comme rayons se croisent en un point T qui est comme on le sait le centre du cercle inscrit au triangle ABC et se trouvera au point milieu de la polaire DE de A par rapport au cercle O′.

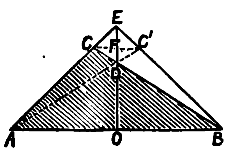
Abaissons de T la perpendiculaire TL = r sur AC ; comme angle

$$O'ET = ETL = \tfrac{1}{2}A, \quad \text{on a} \quad O'E = TE \sec \tfrac{1}{2}A = r \sec^2 \tfrac{1}{2}A.$$

Démonstrations analogues pour les trois autres cercles tangents extérieurement à la circonférence circonscrite dans les angles A, B, et C.

12419. (Professor Morel.)—Dans tout triangle, toute hauteur ϕ est moyenne harmonique entre les deux segments, determinés sur la perpendiculaire au côté correspondant (la médiatrice) à cette hauteur menée par le milieu de ce côté, par les deux autres côtés, ces segments ayant pour origine commune le point milieu.

Solution by Professors Schoute, Bhattacharya, *and others.*



D'après les propriétés connues du quadrilatère complet les deux couples de points (O, F) et (D, E) se séparent harmoniquement l'une l'autre, etc.

12414. (Professor Droz-Farny.)—On donne un point fixe A sur une circonférence O et un point quelconque P. Une circonférence

variable par A et P coupe la première en B et la diamètre OP en C. (1) La droite BC passe par un point fixe; (2) lieu du point d'intersection de BC avec la tangente en A au cercle variable; (3) la tangente en C enveloppe une parabole.

Solution by Professors SCHOUTE, SARKAR, *and others.*

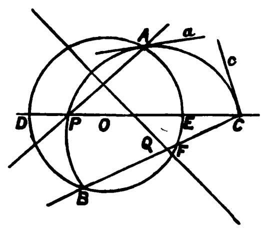


Fig. 1.

(1) Les points B et C décrivent sur la circonférence O et la droite OP des ponctuelles projectives. Donc la droite BC enveloppe une courbe dont la classe est égale à la somme des ordres de la circonférence O et de la droite OP. Parce que les deux points d'intersection D et E (Fig. 1) de la circonférence O et de la droite OP sont des points de coïncidence de B et C, l'enveloppe de la troisième classe dégénère en trois faisceaux de rayons, dont D, E et le point F en question sont les sommets.

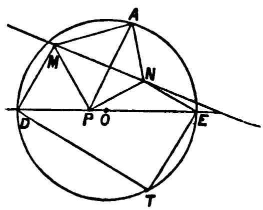


Fig. 2.

On reconnaît sans peine que le point F se trouve sur la circonférence O. En effet, représentons par M et N (Fig. 2) les centres des cercles APD et APE et cherchons les positions correspondantes de la droite BC. En supposant que ces cercles rencontrent la circonférence O et la diamètre OP en des points différents infiniment voisins l'un de l'autre, on trouve que ces positions correspondantes de BC sont les tangentes en D et E à ces cercles. Ces deux tangentes forment un angle droit. Car on a

$$\angle \text{AMP} = 2\text{ADP}, \quad \text{PNA} = 2\text{PEA},$$

$$\angle \text{AMP} + \text{PNA} = 2(\text{ADP} + \text{PEN}) = 180^\circ,$$

donc $$\angle \text{MPN} = 90^\circ, \quad \text{ou} \quad \text{DPM} + \text{NPE} = 90^\circ,$$

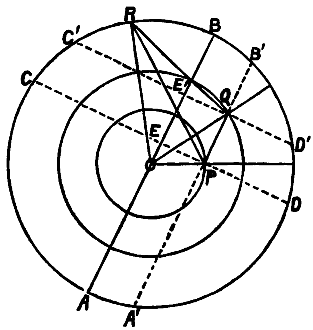
ou $$\angle \text{MDP} + \text{PEN} = 90^\circ.$$

(2) La droite BC (Fig. 1) et la tangente *a* en A au cercle variable BPAC sont des rayons homologues de deux faisceaux projectifs à sommets F et A. Donc le lieu demandé est une conique qui passe par F et A.

(3) Le rayon QC (Fig. 1) du cercle variable BPAC joint des points correspondants Q et C de deux ponctuelles projectives. Parce que les points à l'infini de ces deux ponctuelles se correspondent l'un à l'autre, l'enveloppe est une parabole. Donc on peut dire que QC joint des points correspondants C et R, de deux ponctuelles projectives situées sur OP et sur la droite r_∞ à l'infini. Soit R′ le point de r_∞ situé dans la direction perpendiculaire de R. Alors C et R′ parcourent sur OP et r_∞ des ponctuelles projectives, ce qui prouve que la droite CR′, c'est à dire la tangente *c* en C au cercle BPAC, enveloppe une parabole qui touche OP.

9318. (Elizabeth Blackwood.)—P, Q, R are points taken at random in the circumferences respectively of three concentric circles with radii p, q, r. Required (1) the average area of the triangle PQR, and (2) the chance that it has an obtuse angle.

Solution by H. J. Woodall, A.R.C.S.; Professor Bhattacharya; *and others.*



Taking OP for initial line, let

$$POQ = \theta, \quad QOR = \phi;$$

then the sides of the triangle, and area S, can be immediately obtained. We require the average value

$$\int_0^{2\pi} S\,d\phi \Big/ \int_0^{2\pi} d\phi,$$

then $\int_0^{2\pi}\int_0^{2\pi} S\,d\phi\,d\theta \Big/ 4\pi^2.$

Produce PQ to A′B′, draw diameter AB parallel to PQ, draw DPEC, D′QE′C′ through P, Q, respectively, and perpendicular to PQ to cut AB in E, E′. Now, for an obtuse-angled triangle, R must lie in the arcs DAC, D′B′C′. We have

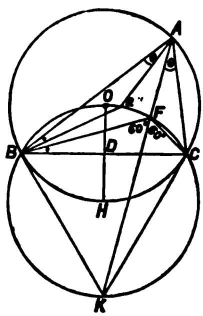
$$\sin COC' = \left[q\,(q-p\cos\theta)\left\{r^2\,(p^2+q^2-2pq\cos\theta)-p^2\,(p-q\cos\theta)^2\right\}^{\frac{1}{2}} \right.$$
$$\left. -p\,(p-q\cos\theta)\left\{r^2\,(p^2+q^2-2pq\cos\theta)-q^2\,(q-p\cos\theta)^2\right\}^{\frac{1}{2}}\right]$$
$$\Big/ \left[r^2\,(p^2+q^2-2pq\cos\theta)\right] = \sin\Theta, \text{ say.}$$

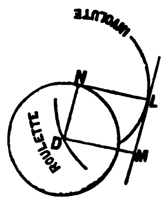
Required chance $= 1-\Theta/\pi$, if θ varies, this becomes

$$= \int_0^{2\pi}\left\{1-\Theta/\pi\right\}d\theta \Big/ \int_0^{2\pi} d\theta = 1-\int_0^{2\pi}\Theta\,d\theta \Big/ 2\pi^2.$$

12284. (R. Chartres.)—BC is a fixed chord of a circle subtending an angle of 120° at the centre O: show that (1) for any position of A on the larger arc the ortho-centre, in-centre, circum-centre, Fermat's point, and another point Z of the triangle ABC lie on the arc BOC; also find (2) Z′, the isogonal conjugate of Z, and the value of $\Sigma(ZA).\Sigma(Z'A)$, without restricting A to the circle; and (3) the locus of the centre of the nine-point circle in (1), and, if the involute of this curve roll on a straight line, find the locus of the mid-point of BC.

Solution by Professor Longchamps; *the* Proposer; *and others.*





BC subtends at the ortho-centre, in-centre, circum-centre, Fermat's point F, and its isogonal conjugate Z, $180^\circ - A$, $90 + \frac{1}{2}A$, $2A$, 120°, and $A + 60^\circ$, each of which $= 120^\circ$, if $A = 60^\circ$.

By the similar triangles AZB, AKC,

$$ZA/c = b/AK,$$

or $\quad ZA \,.\, \Sigma\,(FA) = bc$;

$$\therefore \Sigma(ZA) \,.\, \Sigma(FA) = \Sigma(ab).$$

Since the radius of the nine-point circle $= \frac{1}{2}R = OD =$ constant, therefore the locus of its centre is the circle having ODH for diameter.

If the involute of this circle roll on a straight line LM, then LM equals sub-normal of the roulette described by $D = \frac{1}{2}R =$ constant; therefore D describes a parabola.

9348. (Professor Beni Madhav Sarkar.)—The vertex of a paraboloid of revolution is on a sphere, and the axis of the paraboloid touches the sphere; find the centre of gravity of that portion of the surface and volume of the paraboloid which is enclosed by the sphere.

Solution by H. J. Woodall, A.R.C.S.; Professor Aiyar; *and others.*

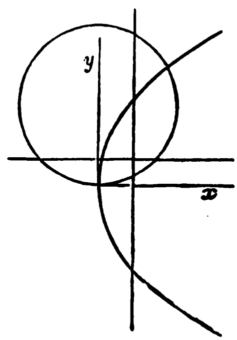


Fig. 1.

The sphere and paraboloid are

$$x^2 + (y-a)^2 + z^2 = a^2,$$

$$y^2 + z^2 = 4bx.$$

Fig. 1 shows the form of the section made by a plane $z = 0$ (z being the vertical axis); the curves are $\quad x^2 + y^2 = 2ay, \quad y^2 = 4bx.$

Both centres of gravity lie in this plane. If (x, y) be point of intersection (other than origin), y is a root of $y^3 + 16b^2y - 32ab^2 = 0$.

Fig. 2 is section by plane $x = h$. In this figure, we have

$$AB = a,\ AC = AC' = (a^2 - h^2)^{\frac{1}{2}},$$

$$BC = BC' = 2\,(bh)^{\frac{1}{2}},$$

$$\cos CAB$$

$$= (2a^2 - h^2 - 4bh)/2a\,(a^2 - h^2)^{\frac{1}{2}},$$

$$\cos CBA = (h^2 + 4bh)/4a\,(bh)^{\frac{1}{2}},$$

$$CO = CB \sin CBA$$

$$= \tfrac{1}{2}\left\{16a^2bh - (h^2 + 4bh)^2\right\}^{\frac{1}{2}}/a.$$

Curved area CC' = sum of sectors

$$ACC' + BCC' - 2\Delta ACB$$

[replace h by x]

$$= \Big\{(a^2 - x^2) \text{ arc cos } \left[(2a^2 - x^2 - 4bx)/2a\,(a^2 - x^2)^{\frac{1}{2}}\right] + 4bx \text{ arc cos}\left[(x^2 + 4bx)/4a(bx)^{\frac{1}{2}}\right]\Big\}/\pi - \tfrac{1}{2}\left\{16a^2bx - (x^2 + 4bx)^2\right\}^{\frac{1}{2}} = A_x, \text{ say.}$$

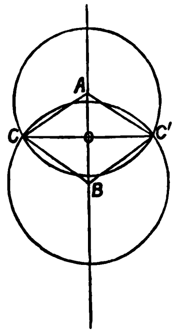


Fig. 2.

Then required

$$(x) = \int_0^{x_1} xA_x\,dx \Big/ \int_0^{x_1} A_x\,dx.$$

If we cut the system by a plane $y = y$, we shall get Fig. 3; the curves are

$$x^2 + z^2 = 2ay - y^2,\quad z^2 = 4bx - y^2,$$

$$OL = y^2/4b = x_1;$$

the curves cut where

$$x^2 + 4bx + 4b^2 = 2ay + 4b^2;$$

$$\therefore \quad x_2 = -2b + (2ay + 4b^2)^{\frac{1}{2}},$$

where we must take the positive sign of the radical.

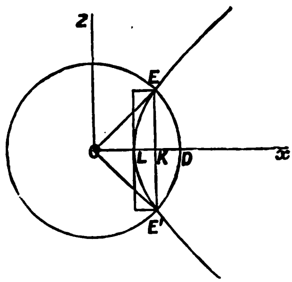


Fig. 3.

$$z_2 = \pm(4bx - y^2)^{\frac{1}{2}} = \pm\left\{4b\,(2ay + 4b^2)^{\frac{1}{2}} - 8b^2 - y^2\right\},$$

$$\text{area of } LKE = \tfrac{2}{3}LK \times KE = \tfrac{2}{3}\,(x_2 - x_1)\,z_2,$$

$$\text{area of sector of circle} = (2ay - y^2) \text{ arc tan } (z_2/x_2);$$

therefore area of curved portion

$$= \tfrac{2}{3}\,(x_2 - x_1)\,z_2 + (2ay - y^2) \text{ arc tan } (z_2/x_2)\,\pi - x_2z_2 = A_y,$$

$$(y) = \int_0^{y_1} yA_y\,dy \Big/ \int_0^{y_1} A_y\,dy.$$

For the C.G. of the included surface, we have

$$(x) = \Sigma\,(\text{arc} \times \text{dist. of C.G. from origin})/\Sigma \text{ arcs},$$

where

$$\text{arc} = CB \times \text{angle } CBC'.$$

But (MINCHIN, *Statics*, Vol. I., p. 272)

$$(x_1) = a \sin \alpha/\alpha = a \times 2a \sin \alpha/(2a\alpha) \; ; \quad \therefore \; (x_1) \times 2a\alpha = 2a^2 \sin \alpha \, ;$$

applying this, we get

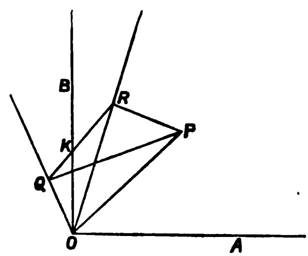
$$(x) = \int_0^{y_1} 8bx \left\{16a^2bx - (x^2+4bx)^2\right\}^{\frac{1}{2}}/4a\,(bx)^{\frac{1}{2}}\,dx$$

$$\Big/ \int_0^{y_1} 4\,(bx)^{\frac{1}{2}} \operatorname{arc\,cos}\left[(x^2+4bx)/4a\,(bx)^{\frac{1}{2}}\right] dx,$$

so with (y) in a similar manner.

12378. (Professor DROZ-FARNY.)—Si d'un point d'une hyperbole équilatère, on abaisse des perpendiculaires sur deux diamètres conjugués, la droite qui joint leurs pieds a une direction constante.

Solution by Professor LAMPE ; H. W. CURJEL, B.A. ; *and others.*



Let PR, PQ be the perpendiculars from P, a point on an equilateral hyperbola with axes OA, OB on the conjugate axes OR, OQ; then OB bisects the $\angle$ ROQ. Let OB cut QR in K. Then

$$\angle \mathrm{POA} + \mathrm{POR} + \mathrm{ROK} = \text{a right angle},$$

and

$$\angle \mathrm{QOK} + \mathrm{PQR} + \mathrm{BKR} = \mathrm{QOK} + \mathrm{PQK} + \mathrm{QKO} = \text{a right angle},$$

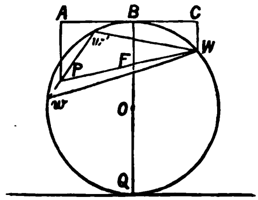
for PQO is a right angle. But $\angle$ RQP = ROP,

and QOK = KOR;

$\therefore$ $\angle$ RKB = POA = a constant.

1080. (EDITOR.)—Three given weights (considered as heavy material points) are attached to the surface of a sphere; find the position of equilibrium of the sphere when resting on a horizontal plane, and give the result in the particular case in which the weights are arranged in a great circle.

Solution by M. Brierley, Professor Chakrivarti, *and others.*



Let OQww'W be the sphere at rest upon Q; W the heaviest weight upon it upon the right side of the vertical plane BOQ through the centre O; and w, w' the two least weights upon the left side of the said plane, and P their centre of gravity within the sphere. Join P, W, cutting BOQ in F; and let ABC be a horizontal tangent to the sphere at B, upon which draw the perpendiculars PA, WC.

Then $\qquad w+w' = \text{P} = \text{W};$

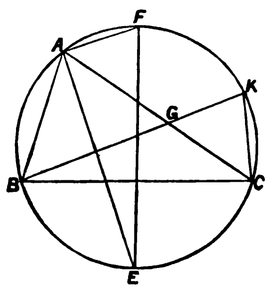
or $$w+w' = \text{PF} \quad \text{and} \quad \text{W} = \text{FW};$$

whence $$\text{W} : w+w' = \text{PF} : \text{FW} = \text{AB} : \text{BC}.$$

When w, w' are arranged in a great circle, the plane of W, w, w' cuts the sphere through the centre O.

12244. (I. Arnold.)—If from the mid-point of the base of a triangle a line be drawn perpendicular to the base cutting the bisector of the exterior angle at the vertex, and if the part intercepted between the vertex and the perpendicular be equal to the difference of the sides, show that the vertical angle is a given angle.

Solution by W. J. Dobbs, M.A.; H. W. Curjel, B.A.; *and others.*



Let the internal and external bisectors of the $\angle$ A of $\triangle$ABC cut the circumcircle in E and F, cut AG from AC = AB, and let BG cut the circumcircle in K. Then EF bisects BC at right angles,

$$\angle \text{AEF} = \angle \text{KBC},$$

$$\text{KC} = \text{AF} = \text{GC},$$

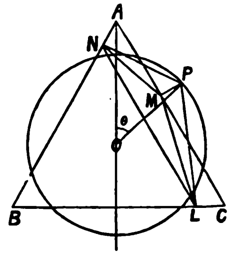
$$\angle \text{CKG} = \text{KGC};$$

hence $\triangle$ABG is equiangular;

$$\therefore \quad \angle \text{BAC} = \tfrac{1}{3}\pi.$$

9675. (Professor Darboux.)—Étant donnés un triangle équilatéral ABC et une circonférence concentrique Δ, les triangles qui ont pour sommets les projections d'un point quelconque de Δ sur les côtés de ABC ont même angle de Brocard.

Solution by H. J. WOODALL, A.R.C.S.



Let $AB = a$, $OP = r$, $\angle AOP = \theta$; then $PL = \frac{1}{3}a + r\cos\theta$, $PM = r\sin(\theta + 30^\circ) - \frac{1}{3}a$;

$$PN = r\sin(\theta - 30^\circ) + \tfrac{1}{3}a,$$

$$MN^2 = a_1^2 = (\tfrac{3}{4}r^2 + \tfrac{1}{3}a^2) - ar\cos\theta;$$

$$NL^2 = b_1^2 = (\tfrac{3}{4}r^2 + \tfrac{1}{3}a^2) + ar(\tfrac{1}{2}\sqrt{3}\sin\theta + \tfrac{1}{2}\cos\theta),$$

$$LM^2 = c_1^2 = (\tfrac{3}{4}r^2 + \tfrac{1}{3}a^2) + ar(-\tfrac{1}{2}\sqrt{3}\sin\theta + \tfrac{1}{2}\cos\theta).$$

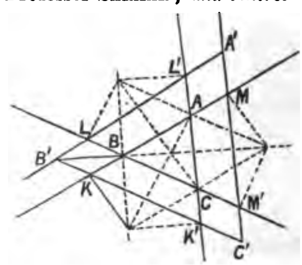
Thus

$$\cos^2 w = \tfrac{1}{4}(\tfrac{9}{4}r^2 + a^2)^2 / \left\{3(\tfrac{3}{4}r^2 + \tfrac{1}{3}a^2)^2 - \tfrac{3}{4}a^2r^2\right\}$$

does not vary with θ, hence such triangles have the same Brocard-angle.

4125. (Prof. HUDSON, M.A.)—If a line join the points of contact of an escribed circle with the produced sides of a triangle, and corresponding lines be drawn for the other escribed circles so as to form an outer triangle; prove that the lines joining corresponding vertices of the two triangles are perpendicular to the sides of the former and that they are equal to the radii of the escribed circles. Also if from the outer triangle another triangle be formed in the same way, and so on, prove that these triangles tend to become equilateral.

Solution by H. J. WOODALL, A.R.C.S.; Professor SARKAR; *and others.*



Let ABC, A′B′C′ be the given and new triangles; the points of contact of the escribed circles be K, K_1, &c.; then in the quadrilateral BLB′K, we have

$$BL = s - a, \quad BK = s - c;$$

angles at B, L, B′, K are B,

$$\tfrac{1}{2}(\pi + C), \quad \tfrac{1}{2}(\pi - B); \quad \tfrac{1}{2}(\pi + A).$$

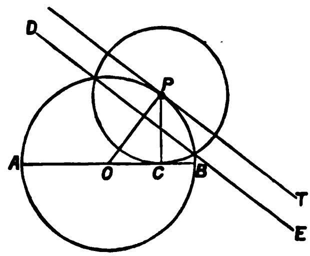
Also $BB' = (s-c)\sin BKB'/\sin BB'K$
$= (s-a)\sin BLB'/\sin BB'L \ldots (1).$

If then we put $KBB' = \phi$ and solve (1) as a linear in $\tan\phi$, we find $\tan\phi = \cot A$, whence $\phi = \frac{1}{2}\pi - A$, and thus BB′ is perpendicular to AC. Then $BB' = (s-c)\sin BKB'/\sin BB'K = (s-c)\cos\frac{1}{2}A/\sin\frac{1}{2}A = S/(s-b)$
= radius of escribed (b) circle.

Again, if A′, B′, C′ be the angles of the second triangle, we find them equal $\frac{1}{2}(B + C)$, $\frac{1}{2}(C + A)$, $\frac{1}{2}(A + B)$, respectively, so also of the third triangle $\frac{1}{4}(\pi + A)$, $\frac{1}{4}(\pi + B)$, $\frac{1}{4}(\pi + C)$, respectively, and so on. Hence the triangles tend to become equilateral.

12122. (Professor DROZ-FARNY.)—On donne une circonférence O et un diamètre fixe AB. D'un point variable P de O comme centre on décrit une circonférence tangente à AB. Soit C le point de contact. On demande (1) l'enveloppe de l'axe radical des circonférences O et P ; (2) le lieu de l'intersection de cet axe radical avec le rayon de contact PC ; et (3) le lieu des points d'intersection de la tangente en P au cercle O avec le cercle P.

Solution by Professors ZERR, BHATTACHARYA, *and others.*



$x^2+y^2 = r^2$ is equation to circle O, $x^2+y^2+a^2 = 2ax+2by$ is equation to circle P, where (a, b) are coordinates to P ; therefore

$2ax+2by = r^2+a^2 =$ equation (1) to radical axis DE.

Also $a^2+b^2 = r^2 =$ a constant (2).

From the first differential equation of (1) and (2), we get

$$b = \sqrt[3]{2r^2}$$

$$a = 2rx/(2r+\sqrt[3]{4ry^2}).$$

These values substituted in one gives for the envelope

$$4r^2(x^2+4y^2-r^2)+4(x^2+y^2-r^2)\sqrt[3]{4ry^2}+6ry\sqrt[3]{2r^2y} = 0.$$

(2) The coordinates of intersection of DE with PC are $x = a,\ y = b/2$; therefore $x^2+4y^2 = a^2+b^2 = r^2$ an ellipse.

(3) Equation to tangent PT is $xa+yb = r^2$.

The coordinates of intersection of this equation with circle P are

$$x = \frac{ar\mp b^2}{r},\quad \text{or}\quad x\pm r = \frac{r\pm a}{r}\,a,\quad y = \frac{r\pm a}{r}\,b\,;$$

therefore $$r^2(x\pm r)^2+r^2y^2 = (r\pm a)^2(a^2+b^2),$$

or $$(x\pm r)^2+y^2 = (r\pm a)^2 \quad\text{.............................(3)},$$

also $$\frac{x\pm r}{y} = \frac{a}{b} \quad\text{.................................(4)};$$

from (4), $$a = \frac{r(x\pm r)}{\{(x\pm r)^2+y^2\}^{\frac{1}{2}}}.$$

Therefore the locus is

$$\left[\{(x\pm r)^2+y^2\}^2-2r^2(x\pm r)^2-r^2y^2\right]^2 = 4r^4(x\pm r)^4+4r^2y^2(x\pm r)^2,$$

or $$\{(x\pm r)^2+y^2\}^4+r^4y^4 = 2r^2\{2(x\pm r)^2+y^2\}\{(x\pm r)^2+y^2\}^2,$$

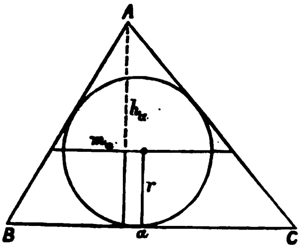
or $$\{(x\pm r)^2+y^2\}^2-r^2y^2 = 2r(x\pm r)^2+y^2\}.$$

The upper sign to be used for one intersection, the lower for the other.

12423. (Professor RUSSO.) — Par le centre du cercle inscrit au triangle ABC, on mène des parallèles aux côtés. Soient m_a, m_b, m_c les parties de ces parallèles comprises entre les côtés. Démontrer que h_a, h_b, h_c désignant les hauteurs du triangle,

$$\frac{m_a}{a}+\frac{m_b}{b}+\frac{m_c}{c}=2, \quad S=\tfrac{1}{4}\left(m_a h_a+m_b h_b+m_c h_c\right).$$

Solution by W. J. GREENSTREET, M.A.; R. CHARTRES; *and others.*



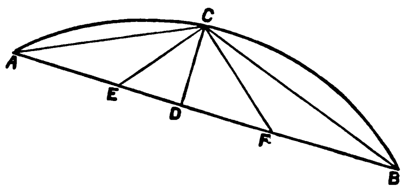
By similar triangles,

$$\Sigma\left(\frac{m_a}{a}\right)=\Sigma\left(1-\frac{r}{h_a}\right)=2,$$

and $\Sigma\left(m_a h_a\right)=\Sigma a\left(h_a-r\right)=4S.$

12432. (I. ARNOLD.)—Given the perimeter of a right-angled triangle and the perpendicular drawn to the hypotenuse from the right angle, construct the triangle.

Solution by M. BRIERLEY; D. BIDDLE; R. CHARTRES; *and others.*



Upon AB, the given perimeter, construct a segment of a circle to contain an angle equal to 135°; in the segment draw CD equal to the given perpendicular; from C draw CE = AE, and CF = FEB; then ECF will clearly be the triangle required.

12365. (R. H. W. WHAPHAM.)—Eliminate λ from the equations

$$a\lambda x-b(1-\lambda)y-a^2\lambda^3+b^2(1-\lambda)^3=0, \quad ax+by-3a^2\lambda^2-3b^2(1-\lambda)^2=0. \quad \text{.........} (1, 2).$$

Solution by the PROPOSER; Professor CHAKRIVARTI; *and others.*

Solving for x, y, we get

$$ax=a^2\lambda^2(3-2\lambda)+2b^2(1-\lambda)^3, \quad by=2a^2\lambda^3+b^2(1-\lambda)^2(1+2\lambda)\text{...}(3, 4);$$

therefore $$ax+by-\frac{3a^2b^2}{a^2+b^2}=\frac{3}{a^2+b^2}\left\{(a^2+b^2)\lambda-b^2\right\}^2 \quad\ldots\ldots\ldots\ldots\ldots(5),$$

and $$ay-bx-\frac{ab(a^2-b^2)}{a^2+b^2}=\frac{2}{ab(a^2+b^2)}\left\{(a^2+b^2)\lambda-b^2\right\}^3\ldots\ldots\ldots\ldots(6).$$

From (5), (6),

$$27a^2b^2\left\{ay-bx-\frac{ab(a^2-b^2)}{a^2+b^2}\right\}^2=4(a^2+b^2)\left\{ax+by-\frac{3a^2b^2}{a^2+b^2}\right\}^3,$$

which is the required eliminant.

11754. (Professor Orchard, M.A., B.Sc.)—If AB be a magnet, and P be a point such that the angles PAB, PBA = 30° and 60° respectively, prove that $\tan^2\theta-\tan^2\theta'=8\sec^2\theta\tan^2\theta'$, where θ, θ' are the angles made by PA, PB, respectively, with a line of force.

Solution by W. J. Greenstreet, M.A.; Prof. Aiyar; *and others.*

If $$\text{PA}=r,\quad \text{PB}=r',$$

$$\sin\theta/\sin\theta'=r^2/r'^2=\sin^2 60°/\sin^2 30=3;$$

therefore $$\sin^2\theta=9\sin^2\theta',$$

or $$\tan^2\theta(1+\tan^2\theta')=9\tan^2\theta'(1+\tan^2\theta),$$

or $$\tan^2\theta-\tan^2\theta'=8\tan^2\theta'(1+\tan^2\theta)=8\tan^2\theta'\sec^2\theta.$$

12355. (J. W. Russell, M.A.)—An amateur gardener buys six border carnations and six fancy carnations. They get mixed, so that he cannot discriminate them. Half-a-dozen at random are placed in the greenhouse, and the rest are planted outside. A fancy carnation will survive the winter in a greenhouse, but the chance that it survives outside is one-third. Each fancy carnation gives three cuttings in the succeeding autumn. Show that he may expect to get a dozen of these cuttings.

Solution by the Proposer.

Follow the chance of any one fancy carnation. The chance that it is put inside is half. It will then survive and produce three cuttings. The chance it is planted out is half. The chance that it will then survive is one-third. It will in that case produce three cuttings. Hence the expectation from one fancy carnation is $\frac{3}{2}+\frac{3}{6}=2$. Hence the whole expectation is twelve cuttings.

12390. (S. Tebay, B.A.)—Find two rational fractions, such that their sum shall be equal to the sum of their squares, which is also a square.

Solution by A. Martin, LL.D.; H. W. Curjel, B.A.; *and others.*

Let x, y be the fractions; then

$$x+y = x^2+y^2, \quad x^2+y^2 = \square \quad \text{......................(1, 2).}$$

Assume $az = x$, $bz = y$; then, from (1),

$$z = \frac{a+b}{a^2+b^2};$$

and, from (2), $a^2+b^2 = \square$, which will have place if $a = p^2-q^2$, $b = 2pq$;

hence $$x = \frac{(p^2-q^2)(p^2+2pq-q^2)}{(p^2+q^2)}, \quad y = \frac{2pq\,(p^2+2pq-q^2)}{(p^2+q^2)},$$

where p and q may have any values, if $p > q$.

Taking $p = 2$, $q = 1$, we have $a = 3$, $b = 4$, and the fractions are $\frac{21}{25}$ and $\frac{28}{25}$.

12312. (W. J. Greenstreet, M.A.)—The locus of the centre of a circle C passing through any point P on a conic S, and the extremities of a diameter, is a conic S′ passing through the origin. The tangent at the origin O is a perpendicular to the symmetric of OP with respect to the axes.

Solution by Professors Droz-Farny, Mukhopadhyay, *and others.*

Soient P′ et P″ les symétriques par rapport aux axes de S du point P. Le triangle P′PP″ étant rectangle en P, le centre du cercle PP′P″ coincide avec O. Sur le diamètre PO il n'y a plus outre O qu'un point du lieu; il suffit de considérer en effet le diamètre AB perpendiculaire à PO. La droite PO ne contenant que deux points du lieu, ce dernier est une conique S′ passant par O.

Au diamètre infiniment voisin de P′P″ correspond un point de S′ infiniment voisin de O et situé sur la perpendiculaire à P′P″ au point O. Il en résulte que cette perpendiculaire est la tangente en O, autrement dit que P′P″ est normale en O à la conique S′.

Il serait facile de démontrer que les axes de S′ sont parallèles à ceux de S et que les asymptotes de S′ sont perpendiculaires aux asymptotes de S.

12347. (W. J. Greenstreet, M.A.)—A circle C passes through a given point P and the points of contact of the tangents from P to an ellipse S, cutting the ellipse again at the points Q, R. Show that the pole P′ of

QR, with respect to S, lies on C; and that P, P′ are concyclic with the foci.

Solution by Professors DROZ-FARNY, BHATTACHARYA, *and others.*

C'est un théorème bien connu que deux points donnés P et P′, ainsi que les quatre points de contact des tangents que l'on peut mener de ces points à une ellipse S, appartiennent à une conique. Si, comme dans l'exemple proposé, cinq de ces points sont sur une circonférence, elle contiendra aussi le sixième.

Soient x_0, y_0 et x', y' les coordonnées des points P, P′; la conique des six points aura pour équation

$$\left(\frac{xx_0}{a^2}+\frac{yy_0}{b^2}-1\right)\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}-1\right)-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_0x_1}{a^2}+\frac{y_0y_1}{b^2}-1\right)=0.$$

Cette équation représente un cercle si

$$x_0y_1+y_0x_1=0,\quad x_0x_1-y_0y_1=c^2.$$

Il en résulte $\qquad x_1=\dfrac{c^2x_0}{x_0^2+y_0^2},\quad y_1=\dfrac{-c^2y_0}{x_0^2+y_0^2}.$

Comme $\qquad \dfrac{x_1}{y_1}=-\dfrac{x_0}{y_0}\quad$ et $\quad (x_1^2+y_1^2)(x_0^2+y_0^2)=c^4.$

Les droites OP et OP′ sont symétriquement disposées par rapport à l'axe des x. Prolongeons OP d'une longueur $O\pi = OP'$; la circonférence qui passe par $P\pi P'$ et qui a son centre sur Oy passera aussi par Q et Q′. Car $OQ \cdot OQ' = OP \cdot O\pi = OP \cdot OP' = -c^2$.

On démontrerait de même que les points P, P′ et les foyers imaginaires de l'ellipse sont concycliques.

10942. (J. D. H. DICKSON, M.A.)—If θ_1, ϕ_1; θ_2, ϕ_2, be a pair of solutions, corresponding to a given value of ψ, of

$$\frac{\cos\theta\sin(\theta-\alpha)}{x}=\frac{\cos\phi\sin(\phi-\beta)}{y}=\frac{\cos\psi\sin(\psi-\beta)}{z},$$

$$x\sin(\phi-\beta)=y\sin(\phi+\alpha),\quad x\sin(\psi-\beta)=z\sin(\psi+\alpha),$$

prove that $\qquad \theta_1+\theta_2+\phi_1+\phi_2=\pi.$

Solution by H. W. CURJEL, B.A.; Professor SARKAR; *and others.*

Eliminating x, y, and z, we get

$$\cos\theta\sin(\theta-\alpha)=\cos\phi\sin(\phi+\alpha)=\cos\psi\sin(\psi+\alpha);$$

$$\therefore\ \sin(2\theta-\alpha)-\sin\alpha=\sin(2\phi+\alpha)+\sin\alpha=\sin(2\psi+\alpha)+\sin\alpha;$$

$$\therefore\ 2\phi+\alpha=2\psi+\alpha\ \text{ or }\ \pi-2\psi-\alpha;\qquad \therefore\ \phi_1+\phi_2=\tfrac{1}{2}\pi-\alpha.$$

$$\sin(2\theta-\alpha)=\sin(2\psi+\alpha)+2\sin\alpha=\sin\gamma\ (\text{say});$$

$$\therefore\ 2\theta-\alpha=\gamma\ \text{ or }\ \pi-\gamma;\quad \therefore\ \theta_1+\theta_2=\tfrac{1}{2}\pi+\alpha;\quad \therefore\ \theta_1+\theta_2+\phi_1+\phi_2=\pi.$$

12374. (Professor HUDSON, M.A.)—The two wheels of a bicycle are 81·68 and 81·07 inches in circumference respectively; how many miles must it go that one wheel may make 100 turns more than the other (*to nearest unit*)?

Solution by T. SAVAGE; W. P. WINTER, B.Sc.; *and others.*

Since in 8107 turns of the larger wheel, the smaller makes 8168 (that is, 61 more), it follows that the required distance is $\frac{100}{61}$ times 8107 turns of the larger wheel, or $17\frac{5837}{43920}$ miles.

2927. (Professor EVANS, M.A.)—Find the probability that

$$\cos\phi_1+\cos\phi_2+\cos\phi_3 > \sqrt{2},$$

where ϕ_1, ϕ_2, ϕ_3 are the angles of an acute-angled triangle.

Solution by R. CHARTRES.

Since ϕ_1, ϕ_2, ϕ_3 are acute angles of a triangle, we have

$$\text{maximum value of } \Sigma(\cos\phi) = 1\tfrac{1}{2} \quad\text{and}\quad \text{minimum} = 1;$$

$\therefore$ probability that $\Sigma(\cos\phi)$ is greater than $\sqrt{2} = 3-2\sqrt{2}$.

12259. (Professor ZERR.)—A sum P is lent at $100r$ per cent. At the end of the first year a payment of x is made; and at the end of each following year a payment is made greater by m per cent. than the preceding payment. If the debt will be paid in n years, show that

$$x = \left\{P(r+1)^n(100)^{n-1}(m-100r\right\}/\left\{(m+100)^n-\left[100(r+1)\right]^n\right\}.$$

If $P = \$10,000$, $100r = 4$, $m = 30$, $x = \$400$, then $n = 9{\cdot}029$ years.

Solution by R. H. W. WHAPHAM, B.A.; Professor CHAKRIVARTI; *and others.*

Let P_k be the amount to be paid after the k^{th} payment; let $\lambda = r+1$, $\mu = (m+100)/100$; then we have

$$P_1 = \lambda\,.\,P-x,\quad P_2 = \lambda\,.\,P_1-\mu x,\quad \ldots,\quad P_{n-1} = \lambda\,.\,P_{n-2}-\mu^{n-2}x,$$

$$P_n = \lambda\,.\,P_{n-1}-\mu^{n-1}x,\quad\text{and}\quad \therefore\ = \lambda^2\,.\,P_{n-2}-(\mu^{n-1}+\lambda\mu^{n-2})\,x = \ldots$$
$$= \lambda^n\,.\,P-(\mu^{n-1}+\lambda\mu^{n-2}+\lambda\mu^{n-3}+\ldots+\lambda^{n-1})\,x;$$

but $P_n = 0$, $\qquad \therefore\ x = P\,.\,\lambda^n(\mu-\lambda)/(\mu^n-\lambda^n)$,

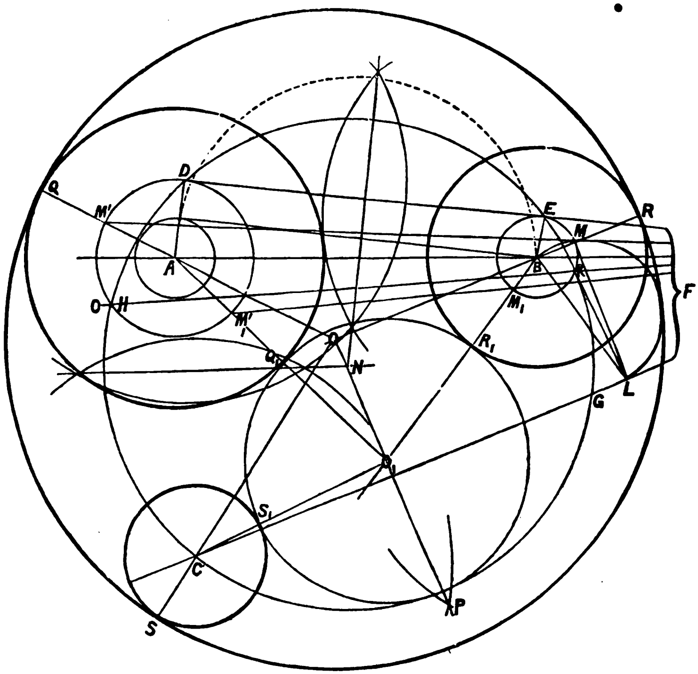
which reduces to result stated on substitution for λ and μ; substituting, we get $400 = 10,000\,.\,104^n.\,26/100\,(130^n-104^n)$, which becomes $(\frac{65}{52})^n = \frac{15}{2}$;

$$\therefore\quad n = \frac{\log 15-\log 2}{\log 65-\log 52} = \frac{{\cdot}8750613}{{\cdot}0969101} = 9{\cdot}029\ldots\text{ years.}$$

12386. (H. J. Woodall, A.R.C.S.)—**Give a geometrical construction for the description of a circle touching three given circles.**

Solution by the Proposer.

Let A, B, C be the centres of the three circles, of radius a, b, c $(a > b > c)$. To find the centre and radius of a circle which shall touch the three given circles. We shall denote a circle by the symbol (a), &c., thus giving both the centre and the radius. With centre A draw $(a-b)$, $(a-c)$; with centre B draw $(b-c)$. The common tangents of $(a-b)$ and $(b-c)$ are found in the usual manner. Take the exterior tangent DE. This will meet AB produced in F; but the two lines generally meet so obliquely that it is better to lengthen AD, BE thus: $AD' : BE' = AD : BE$.



Join D′E′, and produce to meet AB in F. Join CF. About CDE draw a circle, of centre N, to cut CF in G; $(a-c)$ in DH; $(b-c)$ in EK. Join EK, and produce to meet CF in L, from L draw LM tangential to $(b-c)$ and on the other side of B from A, and join BM. Produce BM to meet PN, which bisects CG perpendicularly, in O. OA, OB, OC joined

and produced will meet the circumference in Q, R, S. $OQ = OR = OS$. And O is the centre of the exterior circle required.

Because CDEG is a circle, and DE meets CG in F, therefore $FE . FD = FG . FC$. So again CGKE is a circle; CG meets KE in L; therefore $LK . LE = LG . LC = LM^2$, because LM is tangent to $(b-c)$. Hence the circle CGM touches LM at M, and therefore touches $(b-c)$ at M.

Join FM and produce it to meet $(a-c)$ at M′ and CGM at N′.

Now $FG . FC = FM . FN'$, since CGMN′ is a circle.

Also $FG . FC = FE . FD = FM . FM'$, because F is the centre of similitude of $(a-c)$, $(b-c)$. Therefore $FN' = FM'$, *i.e.*, N′ and M′ coincide.

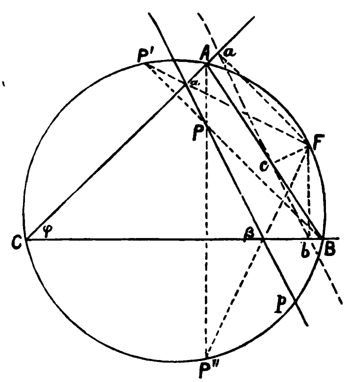
Therefore the circle CGM touches $(a-c)$ at M′. To the radius of this circle CGMM′ add c, and with the same centre O describe the required circle to touch (a), (b), (c).

The modification necessary in order to get the centre of the interior circle touching these circles is easily shown. The points required may, in the majority of cases, be found with fair accuracy, but the method is liable to lead to a mere test of accuracy of your instrument maker.

[In order to simplify the figure, the circle CGM has been omitted.]

12409. (Professor Neuberg.)—On considère toutes les paraboles touchant deux droites données a et b, et dont la directrice passe par un point donné P. Ces courbes ont une troisième tangente commune c, dont on demande une construction. Lorsque P se déplace sur une droite donnée p, la droite c enveloppe une parabole.

Solution by Professors Droz-Farny, Sanjana, *and others.*



Le troisième côté $AB = c$ du triangle admettant a et b comme côtés et P comme orthocentre sera la troisième tangente cherchée d'après le théorème bien connu: L'orthocentre d'un triangle circonscrit à une parabole est sur la directrice de cette dernière. La circonférence circonscrite au triangle ABC passe par les points P′ et P″ symétriques de P par rapport aux côtés a et b.

Menons par P une droite p quelconque coupant a et b en α et β. Les droites $P'\alpha$ et $P'\beta$ symétriques de p par rapport aux côtés a et b se coupent en F sur la circonférence circonscrite. Si P se meut sur p, les droites $P'\alpha$ et $P''\beta$ restent fixes et par conséquent F est un point fixe.

Les pieds a, b, c des perpendiculaires abaissées de F sur les côtés du

triangle ABC appartiennent à la droite de Simson de F, qui est fixe aussi, puisque les points a et b sont invariables.

AB enveloppe donc une parabole ayant F comme foyer et ab comme tangente au sommet.

[Professor SCHOUTE gives the proof thus:—Un triangle étant circonscrit à une parabole, l'orthocentre est sur la directrice (théorème connu). Réciproquement, la droite c, qui joint les points P_a et P où les perpendiculaires abaissées d'un point P de la directrice sur deux tangentes a et b rencontrent b et a, est elle-même tangente de la parabole. Donc la troisième tangente commune à la série de paraboles dont il est question, est indiquée.

Si P parcourt une droite donnée d, les points P_a et P_b parcourent deux ponctuelles semblables, de manière que la droite $c = P_aP_b$ enveloppe une parabole tangente à a et b.]

12430. (H. ORPEUR.) x, y (θ, r) are the Cartesian (polar) coordinates of a point on a curve. The tangent to the curve at that point makes an angle ϕ, (ψ) with the axis of X (radius vector). In each of the following cases, state x, (θ) in terms of ω, and ϕ, (ψ) in terms of ω, and trace the curve. Are there two distinct branches to the curve?—(1) when the sum of the Cartesian (polar) subtangent and subnormal $= 2c$, a constant, and $c \sin\omega = y$ (r), (find the area); (2) when the difference of the Cartesian (polar) subtangent and subnormal $= 2c$, a constant, and $c \tan\omega = y$ (r).

Solution by the PROPOSER; Professor MUKHOPADHYAY; *and others.*

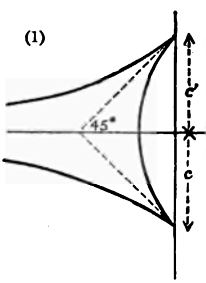
Let P be the point on the curve, O the origin or pole, N the foot of the ordinate, NT the subtangent, NG the subnormal, UO the polar subtangent, OV the polar subnormal.

Let $dx/dy = p$, $d\theta/dr = q$.

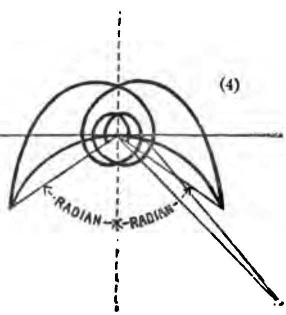
(1) We have $\quad yp^2 - 2cp + y = 0, \quad r^2q^2 - 2cq + 1 = 0,$

in order that $\qquad$ TN + NG $= 2c$, $\quad$ UO + OV $= 2c$.

(1)



(4)



Hence $\qquad y/c = \sin\omega, \quad x/c = \log(1 \mp \cos\omega) \pm \cos\omega$ ....(1),

$$r/c = \sin\omega, \quad \theta = -\cot\tfrac{1}{2}\omega - \omega, \quad -\tan\tfrac{1}{2}\omega + \omega \quad \text{.........(4, 5)},$$

$$\tan\phi = \tan\tfrac{1}{2}\omega, \text{ or } \tan\tfrac{1}{2}(\pi-\omega), \quad \tan\psi = \cot\tfrac{1}{2}\omega \text{ or } \tan\tfrac{1}{2}\omega,$$

$$2\phi = \omega \text{ or } \pi-\omega, \quad 2\psi = \pi-\omega \text{ or } \omega.$$

The entire areas of both these curves is πc^2. In (1) the double sign does not give rise to two branches.

In (4) and (5) one is the reflection of the other about the initial vector.

(2) We have $yp^2 - 2cp - y = 0$, in order that $TN - NG = 2c$,

$$r^2q^2 - 2cq - 1 = 0, \quad ,, \quad ,, \quad UO - OV = 2c.$$

Hence $\qquad y/c = \tan\omega, \quad x/c = \log(\sec\omega \mp 1) \pm \sec\omega$ ....(2, 3),

$$r/c = \tan\omega, \quad \theta = \log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\omega) - \cot\tfrac{1}{2}\omega \text{ or } \log\cot(\tfrac{1}{4}\pi + \tfrac{1}{2}\omega) + \tan\tfrac{1}{2}\omega$$

................(6, 7);

$$2\phi = \omega \text{ or } \pi+\omega; \quad 2\psi = \pi-\omega \text{ or } -\omega.$$

In (2) $y = 0$ is an asymptote. Cuts axis of y at about $(0, \pm 0{\cdot}8\,.\,c)$.

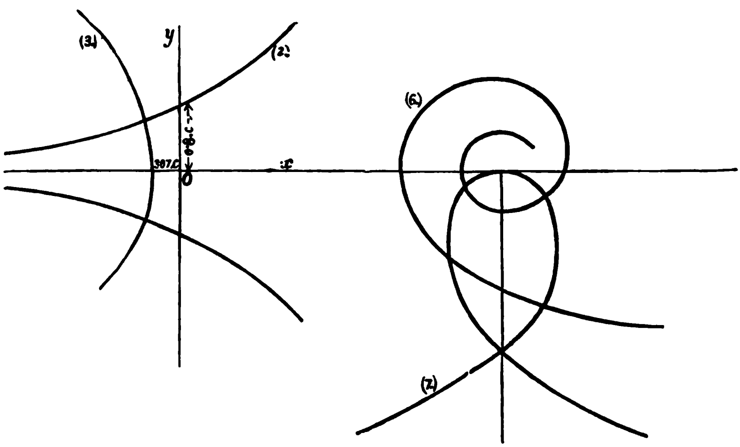
In (3), cuts axis of x at about $(-0{\cdot}307\,.\,c;\ 0)$.

(2) and (3) cut at $(-0{\cdot}43\,.\,c;\ \pm 0{\cdot}66\,.\,c)$, where $\omega = 33^\circ\ 26'$.

(6) is an infinite spiral.

In (7) θ is 0 or $(-\pi)$ or some intermediate value.

The curve is symmetrical about $\theta + \tfrac{1}{2}\pi = 0$.



9878. (Professor Sylvester.)—Every number contains an even number of factors, and therefore the numbers of odd and of even factors are either both odd or both even, except when the original number is a square, and then the reverse is the case.

Solution by H. J. Woodall, A.R.C.S.

Let N be the number, and let its prime factors be a, b, c, ..., so that we have $N = a^{\alpha} b^{\beta} c^{\gamma} \ldots$; then, if T is the sum of all the factors (prime or composite), we have

$$T = (1 + a + a^2 + \ldots + a^{\alpha})(1 + b + b^2 + \ldots + b\ )(1 + c + c^2 + \ldots + c^{\gamma}) \ldots .$$

If $a = 2$, we get the following results:—

No. of odd factors is *odd* or *even* according as $(\beta + 1)(\gamma + 1) \ldots$ is *odd* or *even*,

No. of even factors is *odd* or *even* as $\alpha\, (\beta + 1)(\gamma + 1) \ldots$ is *odd* or *even*.

Hence, if any one of β, γ, ... be odd, the result will be even in both cases, in the second case it will be even if α be even. Accordingly the whole number of factors is odd or even according as α, β, γ, ... are all even or not.

We may sum up thus: If N be square, the number of factors is odd; if $N = 2k^2$, the number of both even and odd factors is odd, the total number of factors is even; if N be not square, the number of both even and odd factors is even, the total number of factors is even.

2036. (Editor.)—A certain sum of money is to be given to the first of three persons A., B., C., who throws 10 with three dice; supposing them to throw in the order named until the event happen, prove that the chances of winning are: A.'s $(8/13)^2$, B.'s $(7 \cdot 8)/13^2$, C.'s $(7/13)^2$.

Solution by Profs. Zerr, Nilkantha Sarkar, *and others*.

The following throws give 10:—631, 622, 532, 541, 442, 433, which can happen in 6, 3, 6, 6, 3, 3 ways respectively, making 27 ways in all; hence the chance of throwing 10 is $27/6^3 = \frac{1}{8}$.

The chances of A., B., C. for the successive throws are

$$\tfrac{1}{8},\ \tfrac{1}{8} \times \tfrac{7}{8},\ \tfrac{1}{8}\,(\tfrac{7}{8})^2;\quad \tfrac{1}{8}\,(\tfrac{7}{8})^3,\ \tfrac{1}{8}\,(\tfrac{7}{8})^4,\ \tfrac{1}{8}\,(\tfrac{7}{8})^5;\ \&\text{c.};$$

hence we have

$$\text{A.'s chance} = \tfrac{1}{8} + \tfrac{1}{8}\,(\tfrac{7}{8})^3 + \tfrac{1}{8}\,(\tfrac{7}{8})^6 + \tfrac{1}{8}\,(\tfrac{7}{8})^9 + \&\text{c. ad inf.} = (\tfrac{8}{13})^2;$$

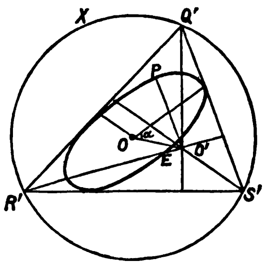
$$\text{B.'s chance} = \tfrac{1}{8}\,(\tfrac{7}{8}) + \tfrac{1}{8}\,(\tfrac{7}{8})^4 + \tfrac{1}{8}\,(\tfrac{7}{8})^7 + \tfrac{1}{8}\,(\tfrac{7}{8})^{10} + \&\text{c. ad inf.} = \frac{7 \cdot 8}{13^2};$$

$$\text{C.'s chance} = \tfrac{1}{8}\,(\tfrac{7}{8})^2 + \tfrac{1}{8}\,(\tfrac{7}{8})^5 + \tfrac{1}{8}\,(\tfrac{7}{8})^8 + \tfrac{1}{8}\,(\tfrac{7}{8})^{11} + \&\text{c. ad inf.} = (\tfrac{7}{13})^2.$$

12253. (W. J. Greenstreet, M.A.)—If through the centre O of an ellipse E, of semi-axes a, b, a straight line be drawn at an angle α to the major axis, from it there be cut off on either side of the centre distances $OD = b$, $OD' = a$, and DOD' be taken as the major axis of a second

ellipse E′, of which O is a focus, prove that (1) one of the common tangents to E, E′ touches E in a point P lying on the auxiliary circle of E′; this circle cuts E in three other points Q, R, S, and the sides of the triangle QRS envelop a fixed circle as α varies; (2) the two ellipses have three other common tangents forming a triangle Q′R′S′, of which the vertices lie on another fixed circle; (3) the perpendiculars of the triangle Q′R′S′ are normal to E and concurrent on the normal at P, cutting it in O′, the second focus of E′; (4) the normals to E at Q, R, S are concurrent in a point ω, and the foot p of the fourth normal through ω lies on the diameter OP; (5) the normals to E at the points of contact of the sides of the triangle Q′R′S′ are concurrent in a point ω' lying on ωp; (6) the locus of ω, as α varies, is an ellipse; (7) the locus of ω', as α varies, is a circle.

Solution by Profs. Ramaswami Aiyar, Chakrivarti, *and others.*



Consider the ellipse E of centre O and semi-axes a, b, and a circle X of centre O and radius $(a+b)$. It may be shown that triangles may be inscribed in X which circumscribe E. Let Q′R′S′ be any such triangle. If O′ be the orthocentre of the triangle Q′R′S′, we may prove that $OO' = (a-b)$ from the well-known theorem that the director circle of any inscribed conic (E) cuts the polar circle orthogonally.

Let the inclination of OO′ to the axis-major of E be α, so that the coordinates of O′ referred to the axis of the ellipse are $(a-b)\cos(-\alpha)$, $(a-b)\sin(-\alpha)$. Let now P be the point on the ellipse, whose eccentric angle is α. It is easy to show what in fact is a well-known result, namely, that PO′ is the normal at P to the ellipse, and PO′ is equal to the semi-diameter conjugate to P.

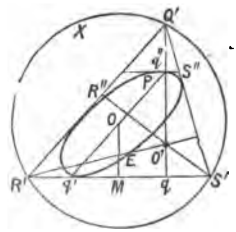
Suppose now an ellipse E′ is inscribed in the triangle Q′R′S′, whose foci are the circumcentre O and orthocentre O′; the axis-major of this ellipse is equal to the circumradius of the triangle Q′R′S′, and is thus equal to $(a+b)$; and, as OO′ is equal to $(a-b)$, we see that the axis-major is divided at either focus, say O, into parts equal to a, b; the square on the minor axis of E′ is equal to ab; hence the tangent to the ellipse E at P is also a tangent to the ellipse E′; and we may also note that P is a point on the auxiliary circle of the ellipse E′.

Conversely, we have the following results: — If E be a given ellipse of centre O and semi-axes a, b, and E′ any ellipse whose focus is O and whose major axis is divided at O into parts equal to a, b, then one of the common tangents to E, E′ touches E at a point P lying on the auxiliary circle of E′, and the other three common tangents form a triangle Q′R′S′ inscribed in a fixed circle X whose centre is O and radius equal to $(a+b)$.

Reciprocating now with respect to a circle whose centre is O and radius squared $= ab$, we may easily deduce that the auxiliary circle of E′ cuts the ellipse E in three other points which are the vertices of a

triangle circumscribing a fixed circle Y whose centre is O and radius equal to $ab/(a+b)$.

Consider again the triangle Q′R′S′ formed by three common tangents to E, E′, and inscribed in X. Let q', q'' be the points of contact of E with the tangent R′S′ and a parallel tangent R″S″. Let $Q'q''$ meet R′S′ in q, and let M be the middle point of R′S′.

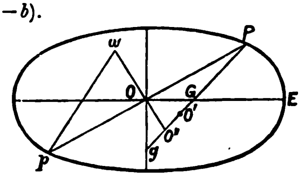


Then, since $q'R \cdot q''R'' = q'S' \cdot q''S''$, each being equal to the square on the semi-diameter of E parallel to R′S′,

$$\therefore \quad q'R' : q'S' = q''S'' : q''R'' = qS' : qR'.$$

Hence M is the middle point of $q'q$; and, as O is the middle point of $q'q''$, it follows that $q''q$ is parallel to OM; therefore $Q'q$ is the perpendicular on R′S′, and is the normal to E at q''. Hence we see that the perpendiculars of the triangle Q′R′S′ are normal to E, and O′ their point of intersection is, as we have seen, on the normal at P, and is the second focus of E′. And we see also that the normals to E at the points of contact of the sides of the triangle Q′R′S′, as well as the normal at the point p diametrically opposite to P on E, are concurrent, the point of concurrence ω' being the symmetrique of O′ with respect to O. It follows that the locus of ω' is the same as that of O′, that is, a circle of centre O and radius $(a-b)$.

Lastly, the auxiliary circle of E′ is the circle whose centre is the middle point of OO′, and which passes through P. By known properties of the Joachimsthal's circle, this circle cuts the ellipse again in three points (Q, R, S), the normals at which are concurrent at a point ω on the normal at p. If O″ be the symmetrique of ω with respect to O, O″ is on the normal PGg, and O′O″ and Gg have the same middle point. Hence $PO' + PO'' = PG + Pg$, and PO′, which is equal to the conjugate diameter, being in a fixed ratio to PG or Pg, whatever P may be, it follows that PO″ is in a constant ratio to the normal PG or Pg; hence the locus of O″ is an ellipse. The locus of ω is the same ellipse, which completes the demonstration.



6510. (Professor Sir R. E. Ball, F.R.S.)—If a rigid body can be rotated about three lines in space, prove (1) that it can be screwed along the three axes of the hyperboloid containing those lines; and (2) show that the pitches of the three screws are inversely proportional to the squares of the axes.

Solution by Frederic R. J. Hervey.

The equations referred to the axes of the hyperboloid of a generating

line which meets the plane xy in the point $(a\cos\theta,\ b\sin\theta)$ are

$$x/a = \cos\theta + z/c\sin\theta,\quad y/b = \sin\theta - z/c\cos\theta.$$

A rotation with angular velocity ω about this line is resolvable into rotations

$$\omega_x = \omega a\sin\theta/\mathrm{A},\quad \omega_y = -\omega b\cos\theta/\mathrm{A},\quad \omega_z = \omega c/\mathrm{A}$$

about the axes, together with translations

$$u = \omega bc\sin\theta/\mathrm{A},\quad v = -\omega ac\cos\theta/\mathrm{A},\quad w = -\omega ab/\mathrm{A}$$

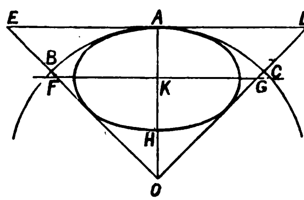
parallel to them, where

$$\mathrm{A} = \sqrt{\left\{a^2\sin^2\theta + b^2\cos^2\theta + c^2\right\}};$$

resolvable, that is, into three screw motions about the axes, whose pitches bc/a, &c., are independent of θ, and proportional to $/a^2$, $/b^2$, $/-c^2$. Hence, by taking three lines of the same system, and compounding rotations about them, the ratios of whose angular velocities satisfy two of the equations $\Sigma\omega_x = 0$, $\Sigma\omega_y = 0$, $\Sigma\omega_z = 0$, we obtain a screw motion about the third axis. [For another solution, see Vol. xxxv., p. 48.]

4984. (Professor EVANS, M.A.)—Find the area of the maximum ellipse that can be inscribed in the quadrant of a given circle.

Solution by Professors ZERR, BHATTACHARYA, *and others.*



The minor axis will coincide with the radius bisecting the quadrant OBAC; also OA = AD = AE = r.

$$x_1^2/a^2 + y_1^2/b^2 = 1$$

is the equation to the ellipse

$$\mathrm{OK} = r - b = b^2/y_1,$$

or $y_1 = b^2/(r-b)$; $\quad$ KG $= a^2/x_1$;

$$x_1 = a^2/(r-b).$$

Substituting these values in the equation to the ellipse, we get

$$a^2 + 2rb = r^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Also $\qquad \pi ab = n = \max. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$

Eliminating the values of da/db obtained from (1) and (2), $a^2 = rb$. This in (1) gives $b = r/3$ and $a = r/\sqrt{3}$; $\therefore\ \pi ab = \pi r^2/3\sqrt{3} = \frac{1}{2}$ (area) of max. ellipse described to touch the semicircle and its diameter symmetrically.

8422. (W. J. GREENSTREET, B.A.)—Trace and find the area of the curves $\qquad r\cos\theta = ae^{-2\operatorname{cosec}^2 2\theta},\quad r\sin\theta = ae^{-2\sec^2 2\theta}.$

Solution by H. J. Woodall, A.R.C.S.

(1) $r\cos\theta = ae^{-2\operatorname{cosec}^2 2\theta}$; the values are—

of θ, $0 \longleftrightarrow \frac{1}{8}\pi \longleftrightarrow \frac{1}{4}\pi \longleftrightarrow \frac{3}{8}\pi \longleftrightarrow \frac{1}{2}\pi$;

of r, $0 +$ increasing $ae^{-4}\sec\theta + ae^{-2}\sqrt{2} + ae^{-4}\sec\theta + \infty$.

then negative till $\theta = \frac{3}{2}\pi$, and so on.

If ϕ be the inclination of tangent to radius vector, we have

$$\tan\phi = rd\theta/dr = \sin^3 2\theta/(1 + 7\cos 2\theta - \cos^2 2\theta + \cos^3 2\theta)$$
$$= \sin^3\theta\cos^3\theta/(\cos^4\theta - \sin^6\theta),$$
$$r = \infty \text{ when } \theta = \tfrac{1}{2}\pi, \quad r = 0 \text{ when } \theta = n\pi.$$

Polar subtangent $= r\tan\phi = 0$ when $r = \infty$;

therefore equation to asymptote is $\theta = \frac{1}{2}(2n+1)\pi$. Actually only the line $\theta = \frac{1}{2}\pi$ needs to be drawn. On tracing the curve, a very great approximation to the asymptote will be found.

The radius vector has a maximum value ($= 0{\cdot}198295a$) when $\theta = 49^\circ 0' 53''$ very nearly (found from $\tan\phi = \infty$, which occurs when $\cos^4\theta - \sin^6\theta = 0$).

(2) $r\sin\theta = ae^{-2\sec^2 2\theta}$.

$$\tan\phi = -\tan\theta\cos^3 2\theta/\{\cos^3 2\theta + 8(1 - \cos 2\theta)\};$$

if $\phi = 0$, $\theta = 0$, if $\phi = \frac{1}{2}\pi$, $\theta = \frac{1}{2}\pi$ or $\cos^3 2\theta + 8(1 - \cos 2\theta) = 0$.

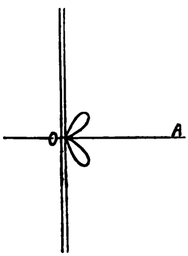
$r = \infty$ if $\theta = 0$ or $= \pi$; $r = 0$ if $\sec^2 2\theta = \infty$, $\theta = \frac{1}{4}(2n+1)\pi$.

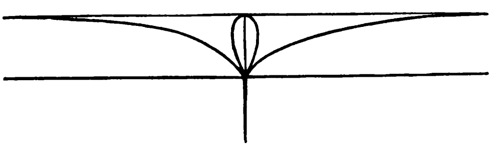
Polar subtangent $= +ae^{-2\sec^2 2\theta}\tan\phi\operatorname{cosec}\theta = ae^{-2}$ when $\theta = 0$.

therefore the asymptote is $r\sin\theta = ae^{-2}$.

The curves cut where $\left.\begin{array}{l}\theta = 23^\circ 56' 29'', \quad r = 0{\cdot}028873a \text{ about} \\ \theta = 69^\circ 12' 49'', \quad r = 0{\cdot}0300116a \text{ about}\end{array}\right\}$.

The curves are difficult to draw, being of the shapes here given.



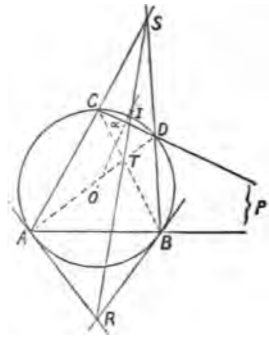


12380. (Professor Fouché.)—On donne un cercle, une corde fixe AB et une corde variable CD de longueur constante. On joint AC, BD qu se coupent en S, puis AD, BC qui se coupent en T. Trouver le lieu décrit par le point d'intersection de la droite ST avec la perpendiculaire élevée au milieu de CD, quand la corde CD se déplace.

Solution by Professor Droz-Farny; H. W. Curjel, B.A.; *and others.*

La corde CD coupe AB en P. Dans le quadrilatère complet ABCD, ST sera la polaire de P, et comme ce point se meut sur AB, ST tourne autour d'un point fixe, le pôle R de AB.

En se déplaçant la corde CD restant de longueur constante son point milieu α décrit une circonférence concentrique à la proposée. La perpendiculaire en α sur CD passe donc constamment par O. Le pôle de CD, devant se trouver sur RST et sur Oα, sera le point d'intersection I de ces deux droites; ce point décrira donc la polaire réciproque de la circonférence, lieu des points α, donc aussi une circonférence de centre O.



12353. (R. Chartres.)—If K be the focal distance of a point O in the axis minor of an ellipse, prove that the maximum straight line OP will be normal to the tangent at P, and, with the usual notation, OP = K/e, OG = Ke, semi-conjugate diameter to CP = bK/ae.

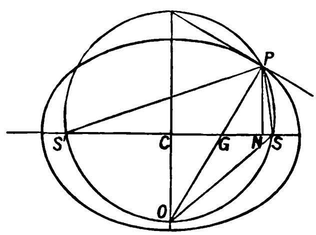
Solution by W. J. Dobbs, M.A.; Professor Ignacio Beyens; *and others.*

By considering two straight lines drawn from O to adjoining points on the ellipse, we see that the maximum OP is normal at P. It is known that the circle round SPS′ passes through O. Hence

$$\angle OSG = OPS' = OPS;$$

therefore triangles OSG and OPS are similar; therefore

$$SO : SG = PO : PS.$$



Hence $PO : SO = SP : SG = 1 : e$;

therefore $PO = K/e$. Also $OG : OP = CG : CN = e^2 : 1$; therefore

$$OG = e^2 . PO = Ke.$$

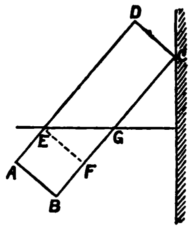
Again, if CD be semi-conjugate to CP, it is known that

$$CD : PG = a : b; \quad \therefore CD = a/b . PG = bK/ae.$$

[This also follows from the Proposer's proof of Quest. 12209.]

4212. (W. Siverly.)—A cylinder of length l and diameter a rests with its upper end against a tank at the side of a vessel containing water in which the lower end floats. The greatest and least lengths of the immersed portion are $b+c$ and b. Find the weight of the cylinder.

Solution by H. J. Woodall, A.R.C.S.



Let the figure be a section by a vertical plane passing through the axis of the cylinder.

Let AD, BF, FG be l, b, c respectively, and

$$AB = 2a.$$

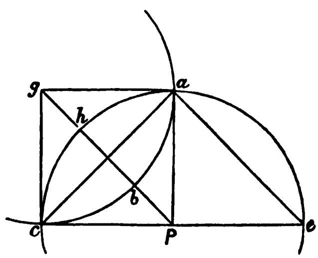
Weight of cylinder

$$= \text{weight of water displaced}$$

$$= \pi a^2 (b + \tfrac{1}{2}c).$$

12273. (Professor Shields.)—A horse is tied to a post P, outside of a circular meadow, with a rope the length of which is equal to the radius of the meadow; find how far from the circumference of the meadow the post must be set to allow the animal to graze over *just one acre* of ground.

Solution by B. T. Brierley; Prof. Ignacio Beyens; *and others.*



ag = radius of meadow $abcf$. aP is perpendicular and equal to ag. The area to be grazed over is evidently $abcdei$, and the distance required is bP. The segments abc and aie are evidently equal to each other; therefore

$$abcdei = \frown ced + \triangle ace$$

$$= 4840 \text{ square yards};$$

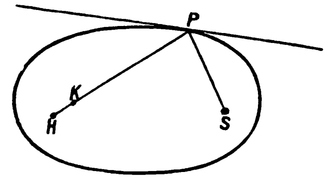
$\therefore$ radius cP $= 43{\cdot}389 +$ yards,

and gP $= 61{\cdot}362 +$ yards.

$$g\text{P} - gp\,(= c\text{P}) = b\text{P} = 17{\cdot}973 + \text{yards}.$$

12162. (Professor Barisien.)—On considère une famille de coniques tangentes à une conique fixe Σ, ayant avec Σ, un foyer commun et passant par le second foyer de Σ. Montrer que toutes ces coniques ont l'axe focal de longueur constante.

Solution by W. J. Dobbs, B.A.; Professor Droz-Farny; *and others.*



Let S be the common focus, H the other focus of Σ. If another conic be drawn having a focus at S and touching Σ at P, its remaining focus must lie in HP (at K, say). Then $\text{SH}+\text{HK} = \text{SP}+\text{PK}$

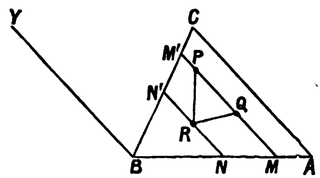
$= \frac{1}{2}$ perimeter of ΔSPH

= a constant; hence

SP + PK is constant; hence the major axis of this second conic is of constant length.

10788 & 11945. (D. Edwardes.)—Prove that (1) the mean area of all triangles having their vertices upon the surface of a given right-angled isosceles triangle of area Δ, is $\frac{13}{210}\Delta$; (2) the same result holds for any given triangle; and (3) the chance that the base angles are acute is $\frac{7}{20}$.

Solution by Profs. Zerr, Mukhopadhyay, *and others.*



(1), (2). Let ABC be any given triangle; P, Q, R the vertices of the triangle whose mean area is required; MPQM′ and RNN′ parallel to AC;

$$\text{BM} = u, \quad \text{BN} = v,$$

$$\text{NR} = z, \ \text{MP} = x, \ \text{MQ} = y,$$

then $\quad mm' = bu/c = x', \quad nn' = bv/c = z',$

$$\text{area PQR} = \tfrac{1}{2}(u-v)(x-y)\sin \text{A} = \beta, \quad v<u;$$

$$\text{area PQR} = \tfrac{1}{2}(v-u)(x-y)\sin \text{A} = \beta_1, \quad v>u.$$

The limits of u are 0, c; of v, 0, u; and u and c; of x, 0 and x'; of y, 0 and x; of z, 0 and z'; hence the required mean area is

$$\text{M} = \int_0^c\int_0^{x'}\int_0^x\left\{\int_0^u \beta\,dv + \int_u^c \beta_1 dv\right\}\int_0^{z'} du\,dx\,dy\,dz \Bigg/ \int_0^c\int_0^{x'}\int_0^x\int_0^c\int_0^{z'} du\,dx\,dy\,dv\,dz$$

$$= \frac{12}{b^3c^2}\int_0^c\int_0^{x'}\int_0^x\left\{\int_0^u \beta\,dv + \int_u^c \beta_1 dv\right\}\int_0^{z'} du\,dx\,dy\,dz$$

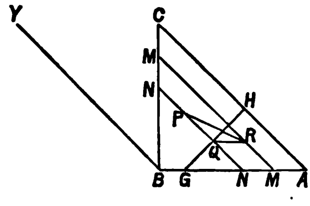
$$= \frac{12}{b^2c^3}\int_0^c\int_0^{x'}\int_0^x\left\{\int_0^u \beta\,dv + \int_u^c \beta_1\,dv\right\} v\,du\,dx\,dy$$

$$= \frac{\sin \text{A}}{b^2c^3}\int_0^c\int_0^{x'}\int_0^x (2u^3+2c^3-3c^2u)(x-y)\,du\,dx\,dy$$

$$= \frac{\sin \text{A}}{2b^2c^3}\int_0^c\int_0^{x'} (2u^3+2c^3-3c^2u)\,x^2\,du\,dx$$

$$= \frac{b\sin \text{A}}{6c^6}\int_0^c (2u^6+2c^3u^3-3c^2u^4)\,du = \frac{13bc\sin \text{A}}{420} = \frac{13}{210}\,\Delta.$$

Since ABC is any given triangle, the result is true for a given right-angled isosceles triangle.



(3) Let ABC (Fig. 2) be the given right-angled isosceles triangle, NN′ the line through PQ parallel to AC. Draw MM′ through R parallel to AC, and GH through Q perpendicular to NN′; then, if R be in the triangle AGH, the angle PQR will be obtuse, and the base angles PRQ, RPQ will be acute.

Let $BN = u$, $BM = v$,

$$NP = x, \quad NQ = y, \quad MR = z, \quad AB = BC = c.$$

The limits of u are 0 and c; of v, 0 and c for the whole triangle, and $u-y\sqrt{2}$, and c for the portion AGH; of x, 0 and $\mu\sqrt{2}$; of y, 0 and x; of z, 0 and $v\sqrt{2}$ for the whole triangle, and 0 and $\frac{1}{2}(v+y\sqrt{2}-u)\sqrt{2} = z'$ for the portion AGH. The required chance is

$$p = \int_0^c\int_0^{u\sqrt{2}}\int_0^x\int_{u-y\sqrt{2}}^c\int_0^{z'} du\,dx\,dy\,dv\,dz \Big/ \int_0^c\int_0^{u\sqrt{2}}\int_0^x\int_0^c\int_0^{v\sqrt{2}} du\,dx\,dy\,dv\,dz$$

$$= \frac{6}{\sqrt{2}\,c^5}\int_0^c\int_0^{u\sqrt{2}}\int_0^x\int_{u-y\sqrt{2}}^c\int_0^{z'} du\,dx\,dy\,dv\,dz$$

$$= \frac{3}{c^5}\int_0^c\int_0^{u\sqrt{2}}\int_0^x\int_{u-y\sqrt{2}}^c (v+y\sqrt{2}-u)\,du\,dx\,dy\,dv$$

$$= \frac{3}{2c^5}\int_0^c\int_0^{u\sqrt{2}}\int_0^x (c^2+2cy\sqrt{2}-2cu+u^2+2y^2-2uy\sqrt{2})\,du\,dx\,dy$$

$$= \frac{1}{2c^5}\int_0^c\int_0^{u\sqrt{2}} (3c^2x+3cx^2\sqrt{2}-6cux+3u^2x+2x^3-3ux^2\sqrt{2})\,du\,dx$$

$$= \frac{1}{2c^5}\int_0^c (3c^2u^2-2cu^3+u^4)\,du = \frac{7}{20}.$$

12165. (EDITOR.)—Solve the two systems of equations

$$x^2+y^2+x+y = 530, \quad xy+x+y = 230; \quad x^2+y^2+y = a, \quad xy+\tfrac{1}{2}x = b \ldots (\alpha, \beta).$$

Solution by GERTRUDE POOLE, B.A.; Dr. A. MARTIN; *and others.*

(α) (1) + (2) × 2 gives $x+y = 30$ or -33.

$x+y = 30$ gives $xy = 200$; therefore $x, y = 20, 10$.

$x+y = -33$ gives $xy = +263$; therefore $x, y = \frac{1}{2}\{-33 \pm (37)^{\frac{1}{2}}\}$.

(β) (3) + (4) × 2 gives $(x+y)^2+(x+y)+\frac{1}{4} = (a+2b+\frac{1}{4})$;

therefore $$x+y=-\tfrac{1}{2}\pm(a+2b+\tfrac{1}{4})^{\frac{1}{2}},$$
$$b=x(y+\tfrac{1}{2})=(y+\tfrac{1}{2})\left\{-y-\tfrac{1}{2}\pm(a+2b+\tfrac{1}{4})^{\frac{1}{2}}\right\};$$
therefore $$(y+\tfrac{1}{2})^2\mp(y+\tfrac{1}{2})(a+2b+\tfrac{1}{4})^{\frac{1}{2}}+\tfrac{1}{4}(a+2b+\tfrac{1}{4})$$
$$=\tfrac{1}{4}(a+2b+\tfrac{1}{4})-b=\tfrac{1}{4}(a-2b+\tfrac{1}{4});$$
therefore $$y+\tfrac{1}{2}=\pm\tfrac{1}{2}(a+2b+\tfrac{1}{4})^{\frac{1}{2}}\pm\tfrac{1}{2}(a-2b+\tfrac{1}{4})^{\frac{1}{2}},$$
$$y=-\tfrac{1}{2}\pm\tfrac{1}{2}(a+2b+\tfrac{1}{4})^{\frac{1}{2}}\pm\tfrac{1}{2}(a-2b+\tfrac{1}{4})^{\frac{1}{2}},$$
$$x=\quad\pm\tfrac{1}{2}(a+2b+\tfrac{1}{4})^{\frac{1}{2}}\mp\tfrac{1}{2}(a-2b+\tfrac{1}{4})^{\frac{1}{2}}.$$

7253. (Professor Cochez.)—Etant donnés deux points A et B distants de d, trouver le plus court chemin de A à B en touchant une circonférence.

Solution by H. J. Woodall, A.R.C.S.

Take as axes of x and y the lines (1) which joins A and B, and (2) bisects AB perpendicularly.

Let $(x-h)^2+(y-k)^2=c^2$ be the circle ....(1).

Then the point where this shortest way touches the circle will be on the ellipse which touches the circle at the point, and whose foci are A, B.

This ellipse will be $x/a^2+y^2/(a^2-a^2)=1$ ....(2),

where a is the unknown major axis. Eliminate y between these equations, and we get
$$(a^2-d^2)^2(x^2-a^2)^2+2(a^2-d^2)a^2(x^2-a^2)\left\{k^2+c^2-(x-h)^2\right\}$$
$$+a^4\left\{k^2-c^2+(x-h)^2\right\}=0,$$
a quartic in x, and a quartic in a^2. Regarded in this light we find that a^2 is a root of the equation
$$a^4-a^2\left[d^2+k^2+c^2-h^2+2hx\pm2k\left\{c^2-(x-h)^2\right\}^{\frac{1}{2}}\right]+d^2x^2=0;$$
whence $$a^2=\tfrac{1}{2}\left[d^2+k^2+c^2-h^2+2hx\pm2k\left\{c^2-(x-h)^2\right\}^{\frac{1}{2}}\right]$$
$$\pm\tfrac{1}{2}\left\{\left[d^2+k^2+c^2+h^2-2hx\pm2k\left\{c^2-(x-h)^2\right\}^{\frac{1}{2}}\right]^2-4d^2x^2\right\}^{\frac{1}{2}}.$$
The first differential of this with regard to x, gives (when equated to zero) the values of x corresponding to turning values of a^2. Whence we obtain those turning values.

8328. (J. W. Russell, M.A.)—If S_n be the sum to n terms of a series which is equal to $\frac{b_1}{a_1+}\frac{b_2}{a_2+}\frac{b_3}{a_3+}\dots\frac{b_n}{a_n}$, show that (1) the most general values of b_n and a_n are
$$b_n=(s_n-s_{n-1})c_n+(s_{n-2}-s_{n-1})c_{n-2},\quad a_n=(s_n-s_{n-2})c_n+(s_{n-1}-s_{n-2})c_{n-1},$$

where $c_1, c_2 \dots c_n$ are arbitrary; and (2) deduce the simplest continued fraction equivalent to $u_1 + u_2x + u_3x^2 + \dots + u_nx^{n-1}$.

Solution by H. J. Woodall, A.R.C.S.

(1) Here $s_n = p_n/q_n = (a_np_{n-1} + b_np_{n-2})/(a_nq_{n-1} + b_nq_{n-2})$,

whence $$s_n - s_{n-1} = -b^nq_{n-2}(s_{n-1} - s_{n-2})/q_n;$$

therefore $b_n = c_n(s_n - s_{n-1}) \div (s_{n-2} - s_{n-1})\,c_{n-2}$ (replacing q's by c's);

similarly $$a_n = c_n(s_{n-2} - s_n) \div (s_{n-2} - s_{n-1})\,c_{n-1},$$

where $c_1, c_2, \dots$ are quite arbitrary.

(2) If $s_n - s_{n-1} = u_nx^{n-1}$, and so on, we find

$$b_n = -c_nu_nx^{n-1}/c_{n-2}u_{n-1}x^{n-2} = -c_nu_nx/c_{n-2}u_{n-1},$$
$$a_n = c_n(u_{n-1} + u_nx)/c_{n-1}u_{n-1},$$

which follow at once from the values of b_n, a_n before obtained.

Putting $c_1 = c_2 = c_3 = \dots = 1$, we get the simplest values, viz.,

$$b_n/a_n = -u_nx/(u_nx + u_{n-1}).$$

12163. (Professor Thiry.)—Chercher le minimum de la bissectrice d'un des angles aigus d'un triangle rectangle dont la hauteur relative à l'hypothénuse est constante.

Solution by Professors Droz-Farny, Matz, *and others*.

Soient BC l'hypothénuse, h la hauteur fixe, et BE la bissectrice de l'angle aigu B; on a $\mathrm{BE} = \mathrm{AB}\sec\frac{1}{2}\mathrm{B} = h\,\mathrm{cosec}\,\mathrm{B}\,.\sec\frac{1}{2}\mathrm{B}$.

Le minimum de la bissectrice aura lieu en même temps que le maximum de la fonction $\mathrm{F} = \sin\mathrm{B}\cos\frac{1}{2}\mathrm{B}$, $\mathrm{F} = 2\sin\frac{1}{2}\mathrm{B}\cos^2\frac{1}{2}\mathrm{B}$,

$$\mathrm{F}^2 = 4\sin^2\tfrac{1}{2}\mathrm{B}\cos^4\tfrac{1}{2}\mathrm{B} = 4\sin^2\tfrac{1}{2}\mathrm{B}(\cos^2\tfrac{1}{2}\mathrm{B})^2.$$

Comme $$\sin^2\tfrac{1}{2}\mathrm{B} + \cos^2\tfrac{1}{2}\mathrm{B} = 1,$$

le maximum de F^2 aura lieu pour

$$\sin^2\tfrac{1}{2}\mathrm{B}\sec^2\tfrac{1}{2}\mathrm{B} = \tfrac{1}{2},\ \text{donc}\ \tan\tfrac{1}{2}\mathrm{B} = \tfrac{1}{2}\sqrt{2}.$$

On trouve d'après cela pour $\mathrm{BE} = \frac{3}{2}h\sqrt{3}$.

8094. (Professor Orchard, B.Sc., M.A.)—Four rods, each weighing two ounces, are hinged together so as to form a square frame of which the diagonals are unstretched elastic strings. If, when the frame is suspended from the end of a diagonal, there is equilibrium when each rod makes an observed angle, α, with the vertical, find the modulus of elasticity.

Solution by H. J. WOODALL, A.R.C.S.

It can be easily seen that the tension in the string is $= 3W = 6$ ounces. Let a be the length of a rod; the natural and stretched lengths of the string are $a\sqrt{2}$ and $2a\cos\alpha$ respectively. If α does not differ much from $45°$, we have $F/\sigma = Ex/l_0$; therefore

$$E\sigma = 6\sqrt{2}/(2\cos\alpha - \sqrt{2})\,g = 12(\sqrt{2}\cos\alpha - 1)\,g,$$

the modulus of elasticity commonly called YOUNG's modulus.

12348. (J. H. GRACE, M.A.)—A system of conics passes through four fixed points A, B, C, D, the circles of curvature at A to two of the conics meet again at right angles in E; prove that the locus of E is a circle.

Solution by H. W. CURJEL, B.A.; Prof. NILKANTHA SARKAR; *and others.*

Take A as origin; let the equations to two of the conics be

$$S \equiv ax^2 + 2hxy + by^2 + x = 0, \quad S' \equiv a'x^2 + 2h'xy + b'y^2 + y = 0.$$

Then, if the circles of curvature of $S + \lambda S' = 0$, $S + \mu S' = 0$ cut at right angles, evidently $\lambda\mu + 1 = 0$.

Hence the centres of curvature (x_1, y_1), (x_2, y_2) are given by

$$x_1 = -\frac{\lambda^2+1}{2\left\{(a+\lambda a')\lambda^2 - 2(h+h'\lambda)\lambda + b + b'\lambda\right\}} = -\frac{\lambda^2+1}{2A_1}, \text{ say,}$$

$$y_1 = -\frac{\lambda(\lambda^2+1)}{2A_1},$$

$$x_2 = -\frac{\lambda(\lambda^2+1)}{2\left\{(a\lambda - a') + 2(h\lambda - h')\lambda + (b\lambda - b')\lambda^2\right\}} = -\frac{\lambda(\lambda^2+1)}{2A_2}, \text{ say,}$$

$$y_2 = \frac{\lambda^2+1}{2A_2}.$$

Coordinates of E are respectively double those of the foot of the perpendicular from A on the join of (x_1, y_1), (x_2, y_2).

$\therefore$ E is given by $x = \dfrac{-(\lambda^2+1)(\lambda A_2 + A_1)}{A_1^2 + A_2^2}$, $y = \dfrac{-(\lambda^2+1)(\lambda A_1 - A_2)}{A_1^2 + A_2^2}$;

therefore
$$x^2 + y^2 = \frac{(\lambda^2+1)^2}{A_1^2 + A_2^2};$$

but $\quad \lambda A_2 + A_1 = b(\lambda^4+1) + (a' + 2h - b')(\lambda^3 - \lambda) + \lambda^2 2(a - 2h')$,

and $\quad \lambda A_1 - A_2 = a'(\lambda^4+1) + (a - 2h' - b)(\lambda^3 - \lambda) + \lambda^2 2(b' - 2h)$,

and $\quad A_1^2 + A_2^2 = (\lambda^2+1)\left[(\lambda^4+1)(a'^2+b^2) + 2(aa' - bb' + 2bh - 2a'h')(\lambda^3 - \lambda) + \lambda^2 2\left\{a'b' + ab + 2(h^2 + h'^2 - h'a - a'h - b'h - bh')\right\}\right]$.

Hence, if l, m, n are determined by the equations

$$l(a' + 2h - b') + m(a - 2h' - b) + 2n(aa' - bb' + 2bh - 2a'h') = 0,$$

$$lb + ma' + n(a'^2 + b^2) = 1,$$

$$l(a - 2h') + m(b' - 2h) + n\left\{(a'b' + ab + 2(h^2 + h'^2 - h'a - ha' - b'h - bh')\right\} = 1.$$

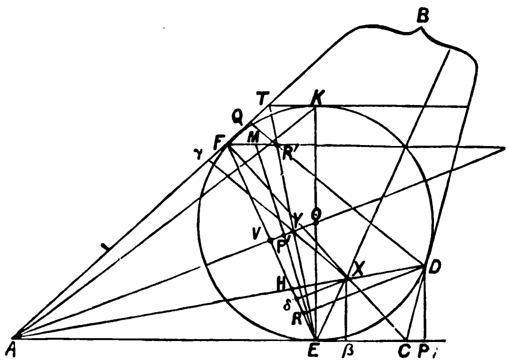
The locus of E is $x^2 + y^2 + lx + my = n$, which is a circle.

12462. (C. E. Hillyer, M.A.)—AB, AC are two fixed tangents to a fixed circle whose centre is O, touching the circle in F and E, and BC is a variable tangent; BE, CF intersect in X. Show that the locus of X is an ellipse whose excentricity is given by the equation

$$\epsilon^2 = (3OE^2 + 3OA^2)/(3OE^2 + 4OA^2).$$

Solution by the Proposer, Professor Mukhopadhyay, *and others.*

Draw $X\beta$, $X\gamma$, $X\delta$ perpendicular to AC, AB, EF respectively. Let BC touch the circle at D; draw DP, DQ, DR perpendicular to AC, AB, EF respectively. Then A, X, D are collinear, since AE, CD, BF are



equal to AF, CE, BD respectively. Let AD meet EF in H. H, D are the intersections of the diagonals of complete quadrilateral AFXE; therefore A, H, X, D form a harmonic range, and

$$\frac{DA}{XA} = \frac{DH}{2XH}.$$

Now $$\frac{DP}{X\beta} = \frac{DA}{XA} = \frac{DQ}{X\gamma};$$

therefore $$\frac{DP \cdot DQ}{X\beta \cdot X\gamma} = \frac{DA^2}{XA^2} = \frac{DH^2}{4XH^2} = \frac{DR^2}{4X\delta^2}.$$

But $DP \cdot DQ = DR^2$, since D is on the circle; therefore $X\beta \cdot X\gamma = 4X\delta^2$; therefore locus of X is a conic touching AB, AC at F and E respectively. Draw TK the tangent to the circle which is parallel to AC, meeting AB in T. Let AK, ET meet in R′; FR′ is parallel to TK or AC, and R′ is a point on the conic; and therefore, if M be the mid-point of FR′, EM is a diameter, and P′ the point where EM meets OA is the centre.

Let ET, EF meet OA in Y and V; EK, ER′, EF, EA form a harmonic pencil; therefore $\dfrac{OY}{YA} = \dfrac{1}{2} \cdot \dfrac{OV}{VA}$.

Also, since M is the mid-point of FR′ (which is parallel to EA), ER′, EM, EF, EA form a harmonic pencil; therefore

$$\frac{OP'}{P'A} = \frac{1}{2}\left(\frac{OY}{YA}+\frac{OV}{VA}\right) = \frac{3}{4}\cdot\frac{OV}{VA}.$$

But
$$\frac{OV}{VA} = \frac{OE^2}{AE^2}; \quad \therefore \frac{OP'}{P'A} = \frac{3OE^2}{4AE^2}.$$

Now, if P′H′ perpendicular to OA meets AB in H′, and if b, a be the semi-axes of the conic, we have

$$\frac{b^2}{a^2} = \frac{P'V \,.\, P'A}{FV \,.\, P'H'} = \frac{P'V \,.\, VA}{FV^2} = \frac{P'V \,.\, VA}{OV \,.\, VA} = \frac{P'V}{OV};$$

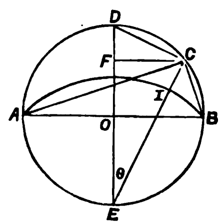
therefore
$$\epsilon^2 = 1-\frac{b^2}{a^2} = \frac{OP'}{OV}.$$

But
$$\frac{OP'}{OA} = \frac{3OE^2}{3OE^2+4AE^2} \text{ and } \frac{OA}{OV} = \frac{OE^2+OA^2}{OE^2};$$

therefore
$$\epsilon^2 = \frac{3OE^2+3OA^2}{3OE^2+4OA^2}.$$

12493. (Morgan Brierley.)—Given the base AB of a triangle ABC, right-angled at C, construct the triangle when the sum of AC and the in-radius is a maximum.

Solution by D. Biddle; W. J. Dobbs, M.A.; *and others.*



Let AB equal the base equal unity. Upon it (as a diameter), describe the circle ADBE, and draw the diameter DE at right angles to AB. Take DF = $\frac{1}{5}$DE, and draw FC parallel to AB, cutting the circle in C. Join AC, BC. Then ABC is the required triangle. For E is the centre of the arc AIB, the locus of the in-centre, and CE bisects the angle ACB, wherever C may be. The angle CED = θ, and in-radius = $\surd\frac{1}{2}\cos\theta-\frac{1}{2}$. Moreover,

$$AC = \sin(\theta+\tfrac{1}{4}\pi) = \surd\tfrac{1}{2}(\sin\theta+\cos\theta).$$

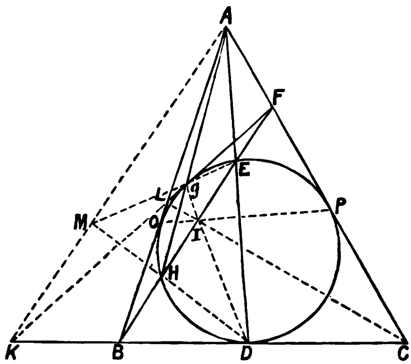
Consequently AC + in-radius = a maximum, when

$$d/d\theta\left\{\surd\tfrac{1}{2}(\sin\theta+\cos\theta)+\surd\tfrac{1}{2}\cos\theta-\tfrac{1}{2}\right\} = 0,$$

that is to say, when $\cos\theta = 2\sin\theta$, or when $\cos^2\theta = 4\sin^2\theta$. Now EF = EC^2 = $\cos^2\theta$, and DF = DC^2 = $\sin^2\theta$, and (by construction) EF = 4DF. Hence, &c. N.B., h = FO = $\frac{3}{10}$AB; also FC = $\frac{2}{5}$AB, and AC = 3BC.

12451. (EDITOR.)—If ABC be a triangle, D the point where BC is touched by the in-circle, AED a straight line cutting the in-circle in E, BHEF a straight line cutting the in-circle in H and AC in F, and FG a tangent from F touching the in-circle in G, prove that A, H, G are in a straight line.

Solution by Professor DROZ-FARNY; W. J. DOBBS, M.A.; *and others.*



La tangente FG coupe AB en L et BC en K. On sait que dans le quadrila'ère circonscrit FLBC les diagonales LC et BF ainsi que les droites GD et OP qui joignent les points de contact des côtés opposés se croisent en un même point T pôle de la droite KA. Dans le quadrilatère GEDH comme les diagonales se croisent en T, les paires de côtés opposés se coupent sur la polaire de T; et par conséquent H, G, et A sont trois points en ligne droite; de même GE, HD, et KA se croisent en un même point M.

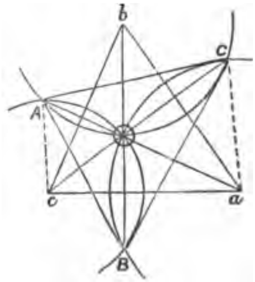
[The theorem is true for any conic inscribed in a triangle.]

12442. (Professor SANJÁNA, M.A.)—A hexagon $AbCaBc$ is such that Aa, Bb, Cc meet in a point O, and

$$cA = cO = cB, \quad aB = aO = aC, \quad bA = bO = bC;$$

prove that O is the orthocentre of abc, and the in-centre of ABC.

Solution by W. J. DOBBS, M.A.; Professor DROZ-FARNY; *and others.*



Aa is common chord of circles with centres b, c, and therefore cuts bc at right angles. Thus aO, bO, cO are respectively perpendicular to bc, ca, and ab, and therefore O is the orthocentre of abc. Again,

$$\angle cAO = cOA = aOC = aCO;$$

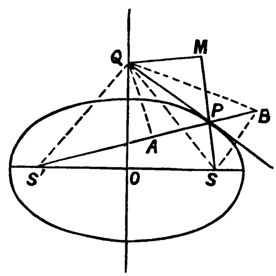
therefore triangles cOA and aOC are equiangular. Hence

$$\angle ABO = \tfrac{1}{2}AcO = \tfrac{1}{2}CaO = CBO.$$

Similarly OA and OC bisect the angles BAC and BCA respectively. Thus O is the in-centre of ABC.

12466. (J. Burke, B.A.)—Let S be a focus of a conic and P any point on the curve, the tangent at which meets the minor axis in Q; let M be the foot of the perpendicular from Q to SP; show that the locus of M is a circle whose centre is S, and whose radius is equal to the semi-major axis of the conic. Hence prove the following method of constructing conics by means of a ruler and compass. Given the two foci S and S′, and the semi-major axis a, with S as centre describe a circle of radius a; let M be any point of this circle, MQ the tangent at M, Q being the point where this line meets the minor axis on the curve. Then the point P in which the circle through SQS′ meets SM is a point on the conic. The method holds for either the ellipse or the hyperbola; in both cases, however, it fails for points very close to the extremities of the major axis.

Solution by Professors Droz-Farny, Sanjána, *and others.*



Soit B le symétrique de S par rapport à la tangente QP. On sait que S′PB est une ligne droite égale à $2a$. Comme QB = QS = QS′, le triangle QS′B est isoscèle et la perpendiculaire QA sur S′B divise cette droite en parties égales S′A = AB = a. De l'égalité des côtés QB et QS et de celle des angles PBQ et PSQ il résulte que les triangles rectangles QMS et QBA sont égaux donc

$$SM = BA = a.$$

La construction sera prouvée si l'on démontre le théorème suivant: Si d'un point Q pris sur le petit axe d'une ellipse on mène des tangentes à cette dernière, le point Q, les points de contact, et les deux foyers sont sur une même circonférence. Or ce théorème est évident car angle QS′B = QBS′ = QSP donc le quadrilatère QS′SP est inscriptible. Même démonstration pour l'hyperbole.

[The Author remarks that the problem admits also of the following analytical solution:—Let (x, y) be the coordinates of M, and (c, O) those of S; then the locus of M comes out to be a quantic which breaks up into the following factors:— $\left[(x-c)^2+y^2\right]\left[(x-c)^2+y^2-a^2\right] = 0$.]

12242. (A. E. Thomas, M.A.)—Solve the equations

$$(2x-y-z)(2y-x-z)(2z-x-y) = 1512, \quad 2(x-y)(x-z) = 234+(y-z),$$
$$7x+4y-2z = 111.$$

Solution by Professor Zerr; R. Chartres; *and others.*

Putting a, b, c for the brackets in (1), we have

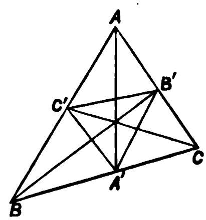
$$a+b+c = 0, \quad abc = 1512, \quad 2(a-b)(a-c) = (b-c)^2+2106,$$

giving $\quad a = 24, \quad b = -21, \quad c = -3.$

Hence, by (3), $\quad x = 17, \quad y = 2, \quad z = 8.$

12453. (Rev. T. C. Simmons, M.A.)—From a ran'om point within a triangle perpendiculars are drawn on the sides; prove that the chance that these can form a triangle is $2abc/\{(a+b)(b+c)(c+a)\}$.

Solution by Rev. T. R. Terry, M.A.; Prof. Sanjána, M.A.; *and others.*



Let the equations of the sides be $\alpha = 0$, $\beta = 0$, $\gamma = 0$. Draw AA′, BB′, CC′, bisecting the angles. Then B′C′ is $\beta+\gamma-\alpha = 0$, C′A′ is $\gamma+\alpha-\beta = 0$, and A′B′ is $\alpha+\beta-\gamma = 0$.

Hence, for points between B′C′ and A, $\beta+\gamma$ is $< \alpha$, and no triangle can be formed by the perpendiculars. So for points between C′A′ and B, and between A′B′ and C.

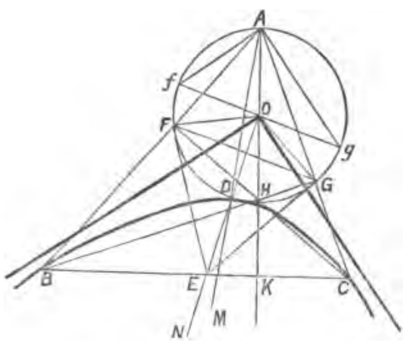
Thus the required chance

$$= \text{area A}'\text{B}'\text{C}'/\text{area ABC}$$

$$= 2abc/\{(a+b)(b+c)(c+a)\}.$$

12473. (Professor Neuberg.)—On donne le sommet A d'un triangle ABC, l'orthocentre H, et la direction de la bissectrice de l'angle BAC. Trouver le lieu décrit par les sommets B et C.

Solution by D. Biddle; H. W. Curjel, M.A.; *and others.*



Join AH and produce. BC will always be at right angles with this line. From O, the mid-point of AH, describe the circle AFHG, and let AM, cutting the circle in D, be the bisector of the angle BAC. Through D draw the straight line ON. This is the locus of the mid-point of BC; for F, G the points in which AB, AC cut the circle are always equidistant from D, and ON bisects and is perpendicular to FG. The positions of B, C are determined by GH, FH, respectively produced. Since BFC, BGC are right-angled triangles having a common base, B, F, G, C are concyclic, and we have

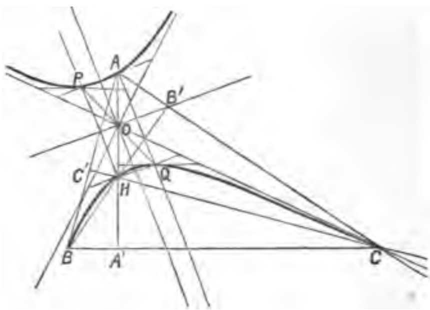
$$\text{EF} = \text{EG} = \text{EB} = \text{EC}.$$

Moreover, since AFH, AGH are right angles as well as AKB, AKC, we have
$$\text{OHF} = \text{OFH} = \text{ABC} = \text{EFB};$$

therefore $\qquad$ EFO = BFC = a right angle.
Therefore EF is always tangential to the circle AFHG, and

$$OE^2 - OD^2 = EF^2 = EB^2.$$

That is to say, $x^2 - a^2 = y^2$, and the required locus is a hyperbola. Also Af, Ag (at the limit when OE is infinite) are separated by a right angle, and the asymptotes are parallel to them; therefore the hyperbola is rectangular.



[Professor Schoute gives the solution thus:—The locus of B (and C) is the locus of the point of intersection of the corresponding rays AB and HB (AC and HC) of two homographic pencils, and therefore a conic passing through A and H.

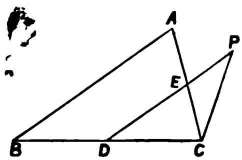
If the angle between AB and the bissectrix is 45°, then AB and HB are parallel. So the conic is a rectangular hyperbola.

The points B and C determine an involution on this hyperbola; the line BC has a fixed direction and passes therefore through a fixed point at infinity, the centre of the involution.

If the angle between AB and the bissectrix is 90° (or nought) the points B and C coincide in P (and Q). So the tangents in P and Q to the hyperbola are perpendiculars to AH. The centre O of the rectangle APHQ is the centre of the hyperbola, the axes of which are parallel and perpendicular to the bissectrix, &c.]

12482. (I. Arnold.)—Given the base BC of a triangle and the sum of the sides AB, AC, find the locus of the intersection of two lines, one drawn from the mid-point D of BC, parallel to AB, the other from C, parallel to the bisector of the vertical angle.

Solution by H. W. Curjel, M.A.; D. Biddle; *and others.*



Let the two straight lines cut in P, and let PD cut AC in E. Then

$$EP = EC = \tfrac{1}{2}AC, \text{ and } DE = \tfrac{1}{2}AB;$$

therefore $\qquad DP = \tfrac{1}{2}(AB + AC)$
$\qquad\qquad\qquad = \text{a constant};$

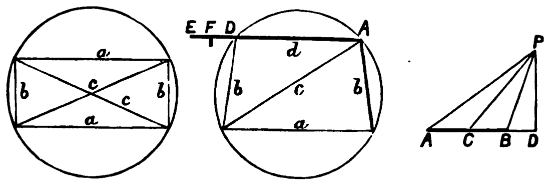
therefore locus of P is the auxiliary circle of the ellipse, which is the locus of A.

11774. (R. CHARTRES.)—Show that (1) Pythagoras' Theorem is only a particular case of Ptolemy's; (2) the difference of the squares of two sides of a triangle equals the rectangle of the base and the parallel chord through the vertex of the circumscribed circle (embracing Euc. II. 12, 13); (3) deduce Euc. II. 9, 10; and find (4) the limit of the ratio of the difference of the sides to the chord in (2) when the sides become ultimately equal.

Solution by Professor AIYAR, *the* PROPOSER, *and others.*

(1) Let the opposite sides of the quadrilateral be equal; then the figure is a right-angled parallelogram, and $a^2+b^2=c^2$ by Ptolemy's theorem.

(2) Let one pair only of opposite sides be equal; then d is parallel to a,



and the diagonals are equal. (i.) Therefore $c^2=b^2+ad$. Now, if $AE=a$, and DE be bisected at F, then $ad=d^2+2$ rectangle AD, DF. (ii.) Or $c^2=b^2+d^2+2\,.\,d\,.\,DF$, which is (II. 12), &c.

(3) Again (by II. 12 and 13), $PA^2+PB^2=2AC^2+2CP^2$. Let P descend vertically to D; therefore $AD^2+BD^2=2AC^2+2CD^2$, which is (II. 9 and 10).

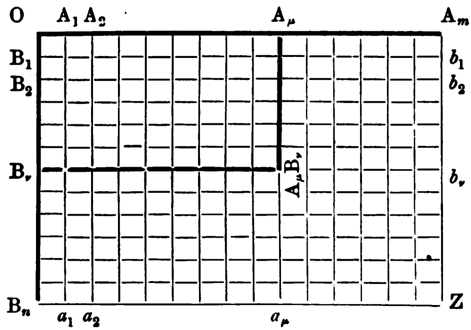
(4) From (2 i.), $\dfrac{c-b}{d}=\dfrac{a}{c+b}$. Let $c-b$, and d become indefinitely small; therefore limit of $\dfrac{c-b}{d}=\dfrac{a}{\text{sum of two equal sides}}$.

12481. (S. ANDRADE, B.A.)—If $f(m,n)$ denote $(m+n)!/(m!\,n!)$, and m, n, μ, ν are positive integers, $m>\mu$ and $n>\nu$, prove that

$$f(m,n)=f(\mu,\nu)\times f(m-\mu,\,n-\nu)+\Sigma_{r=\nu}^{r=1}f(\mu,\,\nu-r)\times f(m-\mu-1,\,n-\nu+r)$$
$$+\Sigma_{r=\mu}^{r=1}f(\mu-r,\,\nu)\times f(m-\mu+r,\,n-\nu-1).$$

Solution by the PROPOSER, Professor BHATTACHARYA, *and others.*

Let OB_n, A_1a_1, $A_2a_2 \dots A_mZ$ be $(m+1)$ parallel streets, and let OA_m, B_1b_1, $B_2b_2 \dots B_nZ$ be $(n+1)$ parallel streets crossing the former, and let $(A_\mu B_\nu)$ denote the intersection of the streets $A_\mu a_\mu$ and $B_\nu b_\nu$.



The number of ways of walking along these streets from O to Z, always tending to one's destination, is $(m+n)!\,/\,(m!\,n!)$, which we denote by $f(mn)$. (See C. Smith's "Algebra," Chapter on Permutations and Combinations.)

Consider the parallelogram O, A_μ, $(A_\mu B_\nu)$, B_ν, and let "nowf" denote "number of ways from."

nowf O to Z = nowf O to $(A_\mu B_\nu)$ × nowf $(A_\mu B_\nu)$ to Z

+ nowf O to $(A_\mu B_{\nu-1})$ × nowf $(A_\mu B_{\nu-1})$ to Z *without going through* $(A_\mu B_\nu)$
+ nowf O to $A_\mu B_{\nu-2}$ × nowf $A_\mu B_{\nu-2}$ to Z " " " $A_\mu B_{\nu-1}$
⋮ &c. for ν terms, going along the line A_μ $(A_\mu B_\nu)$.

+ nowf O to $A_{\mu-1}B_\nu$ × nowf $A_{\mu-1}B_\nu$ to Z without going through $A_\mu B_\nu$
⋮ &c. for μ terms, going along the line B_ν $(A_\mu B_\nu)$.

Since there are only two ways of going at any one point for the moment; therefore nowf $(A_\mu B_{\nu-1})$ to Z without going through $(A_\mu B_\nu)$

= nowf $(A_{\mu+1}B_{\nu-1})$ to Z;

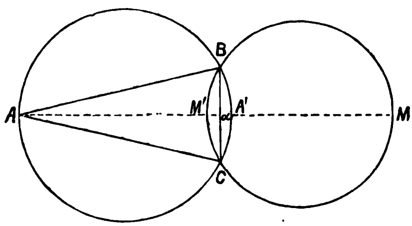
$$\begin{aligned}\therefore f(m, n) = f(\mu, \nu) \times f(m-\mu, n-\nu) &+ f(\mu, \nu-1) \times f(m-\mu-1, n-\nu+1)\\ &+ f(\mu, \nu-2) \times f(m-\mu-1, n-\nu+2)\\ &\;\vdots\\ &+ f(\mu-1, \nu) \times (m-\mu+1, n-\nu-1)\\ &+ f(\mu-2, \nu) \times f(m-\mu+2, n-\nu-1),\\ &\;\vdots\end{aligned}$$

$$i.e., f(m,n) = f(\mu,\nu) \times f(m-\mu, n-\nu) + \sum_{r=\nu}^{r=1} f(\mu, \nu-r) \times f(m-\mu-1, n-\nu+r)$$
$$+ \sum_{r=\mu}^{r=1} f(\mu-r, \nu) \times f(m-\mu+r, n-\nu-1).$$

12059. (Professor Bernès.)—M étant un point quelconque pris dans le plan d'un triangle ABC, démontrer que les puissances de A, B, C rela-

tivement aux circonférences circonscrites à MBC, MAC, MAB et la puissance de M relativement à la circonférence circonscrite à ABC sont inversement proportionnelles aux aires des triangles MBC, MCA, MAB, ABC. Montrer (en tenant compte des signes) que la somme des inverses de ces quatre puissances est nulle.

Solution by Professors Droz-Farny, Chakrivarti, *and others.*



Représentons par A, B, C, M les valeurs des quatre puissances; la droite AM coupe les circonférences circonscrites aux triangles ABC et MBC en A′ et M′ et leur axe radical BC en α.

On a donc
$$\frac{M}{A} = \frac{MA'.MA}{AM'.AM} = -\frac{MA'}{AM'}.$$

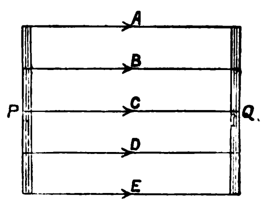
$$\alpha M.\alpha M' = \alpha A.\alpha A'; \quad \frac{\alpha M}{\alpha A} = \frac{\alpha A'}{\alpha M'} = \frac{\alpha M - \alpha A'}{\alpha A - \alpha M'} = \frac{A'M}{M'A};$$

$$\frac{M}{A} = -\frac{\alpha M}{\alpha A} = -\frac{\Delta MBC}{\Delta ABC};$$

il en résulte $\dfrac{M}{A} + \dfrac{M}{B} + \dfrac{M}{C} = -1; \quad \dfrac{1}{A} + \dfrac{1}{B} + \dfrac{1}{C} + \dfrac{1}{M} = 0.$

11585. (Professor Zerr.)—Wires of five different metals A, B, C, D, E, having resistances a, b, c, d, e, have their ends soldered together at two junctions which are maintained at different constant temperatures. If the strength of current in E, when all five wires are continuous, is S, the strength of current when B, C, D are cut is S_a, the strength of current when A, C, D are cut is S_b, the strength of current when A, B, D are cut is S_c, find the strength of current S_x when A, B, C are cut.

Solution by the Proposer.



Let β, γ, δ, λ, ω, μ be the potentials of the wires and solder at one junction P; β', γ', δ', λ', ω', μ' the potential of wires and solder at the other junction Q; v, w, x, y, S the currents in the wires supposed to be going from P to Q.

The electromotive force in the wire A is $(\beta'-\mu')-(\beta-\mu)$, and by Ohm's law this is equal to the product of the resistance into the current.

Hence $(\beta'-\mu')-(\beta-\mu) = av$, or $\beta'-\beta-av = \mu'-\mu$.

Similarly, $\gamma'-\gamma-bw = \mu'-\mu, \quad \delta'-\delta-cx = \mu'-\mu,$
$\lambda'-\lambda-dy = \mu'-\mu, \quad \omega'-\omega-eS = \mu'-\mu.$

By symmetry, $v+w+x+y+S = 0$; similarly,

$$\beta'-\beta+aS_a = \omega'-\omega-eS_a, \quad \gamma'-\gamma+bS_b = \omega'-\omega-eS_b,$$
$$\delta'-\delta+cS_c = \omega'-\omega-eS_c, \quad \lambda'-\lambda+dS_x = \omega'-\omega-eS_x;$$
$$\therefore \quad eS-av = (a+e)S_a, \quad eS-bw = (b+e)S_b,$$
$$eS-cx = (c+e)S_c, \quad eS-dy = (d+e)S_x;$$

$$\therefore \quad (bcde+acde+abde+abce)S-abcd(v+w+x+y)$$
$$= bcd(a+e)S_a+acd(b+e)S_b+abd(c+e)S\ \ +abc(d+e)S_x,$$

or $$S = \frac{bcd(a+e)S_a+acd(b+e)S_b+abd(c+e)S_c+abc(d+e)S_x}{abcd+abce+abde+acde+bcde};$$

whence S_x is known.

12420. (Professor IGNACIO BEYENS.)—Si, dans le plan d'un triangle rectangle, on mène par le sommet de l'angle droit une transversale quelconque, et par chacun des trois sommets, on mène dans le même sens de rotation, des droites faisant chacune avec cette transversale un angle égal à l'angle du triangle correspondant à ce sommet, ces trois droites sont concourantes.

Solution by T. SAVAGE; Professor CHAKRIVARTI; *and others.*

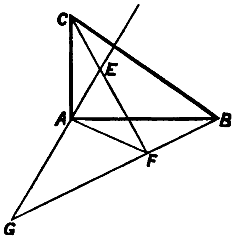
Let the lines AF, CE, fulfilling the conditions, meet in F. Join BF, and produce to meet the secant in G; then

$$\angle AEF = ACB;$$

therefore $$\angle AFE = ABC;$$

hence AFBC is a cyclic quadrilateral; therefore $\angle$ CFG is right; therefore

$$\angle G = B.$$



9646. (W. J. C. SHARP, M.A.)—If a, b, c be the sides of a spherical triangle, R the radius of the sphere, and R_1 that of another sphere through the vertices of the triangle and the centre of the sphere, prove that

$$(4R_1^2-R^2)/R^2$$
$$=2(1-\cos a)(1-\cos b)(1-\cos c)/(1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c).$$

Solution by H. J. WOODALL, A.R.C.S.

Consider the four points A, B, C, O (*i.e.*, the vertices of the triangle and the centre of the sphere) as the vertices of a tetrahedron whose sides

are R, R, R, 2R sin $\frac{1}{2}a$, 2R sin $\frac{1}{2}b$, 2R sin $\frac{1}{2}c$. Then, if V be the volume of this tetrahedron,

$$36V^2 = R^6(1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c)$$

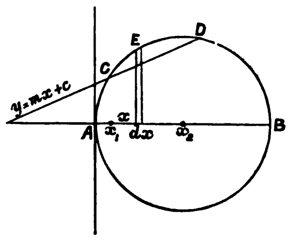
(TODHUNTER's "Spherical Trigonometry," p. 118).

Also $$4\times 144V^2R_1^2 = 16R^8(2\Sigma\sin^2\tfrac{1}{2}b\sin^2\tfrac{1}{2}c-\Sigma\sin^4\tfrac{1}{2}a)$$
$$= 4R^8\left\{2\Sigma(1-\cos b)(1-\cos c)-\Sigma(1-\cos a)^2\right\};$$

therefore, as given, $$(4R_1^2-R^2)/R^2$$
$$=2(1-\cos a)(1-\cos b)(1-\cos c)/(1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c).$$

12418. (Professor DRAUGHTON.)—Find the volume generated by revolving a circular segment, whose base is a given chord, about any diameter as an axis.

Solution by Professor SANJANA; R. CHARTRES; *and others.*



The volume generated by the revolution of the segment CED round AB

$$= \pi\int_{x_1}^{x_2}\left\{2ax-x^2-(mx+c)^2\right\}dx,$$

and is therefore known when the position, &c., of CD is given.

12275. (EDITOR.)—If a, b, c, Δ be the sides and area of a triangle, solve the equation $2\Delta x^4-2abcx^3+(a^2+b^2+c^2)\Delta x^2 = 2\Delta^3$.

Solution by Professor DROZ-FARNY; MORGAN BRIERLEY; *and others.*

On sait que $$r'+r''+r'''-r = 4R = abc/\Delta,$$

$$r = \frac{\Delta}{p},\quad r' = \frac{\Delta}{p-a},\quad r'' = \frac{\Delta}{p-b},\quad r''' = \frac{\Delta}{p-c};$$

donc $$-r(r'+r''+r''')+r'r''+r'r'''+r''r''' = \Sigma p(p-a)-\Sigma(p-a)(p-b)$$
$$= 2p^2-ab-ac-bc = \tfrac{1}{2}(a^2+b^2+c^2).$$
$$1/r'+1/r''+1/r'''-1/r = 0;$$

donc $$rr'r''+rr'r'''+rr''r'''-r'r''r''' = 0.$$

$$-r\,.\,r'\,.\,r''\,.\,r''' = \frac{-\Delta^4}{p(p-a)(p-b)(p-c)} = -\Delta^2.$$

Les racines de l'équation proposée sont donc $-r, r', r'', r'''$.

11410. (Professor DE LONGCHAMPS.)—On considère une hyperbole H, de centre O. Démontrer qu'on peut trouver un cercle Δ, concentrique à H, tel qu'il existe des losanges inscrits à H et circonscrits à Δ. La circonférence Δ est réelle quand l'angle des asymptotes qui contient H est obtus; son rayon s'obtient en élevant une perpendiculaire, au point O, à l'une des asymptotes de H, jusqu'à sa rencontre avec la courbe. Démontrer que si, d'un point de H, on mène les tangentes à Δ, les rayons qui aboutissent aux points de contact forment un faisceau harmonique avec les perpendiculaires élevées, en O, aux asymptotes de H.

Solution by W. J. GREENSTREET, M.A.; Prof. MOREL; *and others.*

Let ABDC be any rhombus inscribed in an hyperbola. Then (1) the chord AC is parallel to the chord BD, and their mid-point join is their conjugate diameter, and passes through the centre of the hyperbola, and through the intersection of AD, BC; (2) the same holds with the chords AB, CD.

Hence the intersection of the diagonals is the centre of the hyperbola, and we know that the diagonals of a rhombus intersect at right angles.

If $x\cos\alpha+y\sin\alpha-p=0$ be one of the sides AC of the rhombus, and O be the centre of the hyperbola, the equation to OA, OC will be

$$(b^2x^2-a^2y^2)\,p^2-a^2b^2\,(x\cos\alpha+y\sin\alpha)^2=0.$$

But, as $\angle AOC=90^\circ,\quad p^2\,(b^2-a^2)=a^2b^2$;

therefore p is the radius of a circle concentric with the hyperbola, and inscribed in the rhombus.

There are two solutions if $b^2>a^2$, *i.e.*, if $b/a>1$, or if $\tan\frac{1}{2}\theta>1$ (where θ is the angle between the asymptotes), *i.e.*, if $\theta>90^\circ$. If the hyperbola be equilateral, there is only one solution.

The line $by-a=0$ perpendicular to one of the asymptotes meets the curve in $\pm\dfrac{ab^2}{(b^4-a^4)^{\frac{1}{2}}},\ \pm\dfrac{a^2b}{(b^4-a^4)^{\frac{1}{2}}}$, the points M, N (say). Then

$$MO=\frac{ab}{(b^2-a^2)^{\frac{1}{2}}}=p.$$

Lastly, the polar of P (α,β) on the hyperbola, with respect to the circle

$$x^2+y^2=\frac{a^2b^2}{b^2-a^2},\quad\text{is}\quad \alpha x+\beta y=\frac{a^2b^2}{b^2-a^2},$$

and the equation of joins of the origin to the contact points of tangents from P to the circle is $a^2b^2\,(x^2+y^2)=(b^2-a^2)\,(\alpha x+\beta y)^2$. These form an harmonic pencil if $b^2\alpha^2-a^2\beta^2-a^2b^2=0$, which is the condition that $\alpha\beta$ is on the hyperbola.

12025. (Professor DÉPREZ.) — On donne deux circonférences de rayons R, r. Quelle doit être la distance des deux centres pour qu'on puisse décrire une circonférence touchant les circonférences données, leur ligne des centres et une tangente commune?

Solution by H. J. WOODALL, A.R.C.S.; Professor SARKAR; *and others.*

Let D be the distance between the two centres, ρ the new radius.

The condition that the new circle should touch the line of centres as well as the two given circles is thus

$$D = \{(R+\rho)^2-\rho^2\}^{\frac{1}{2}} + \{(r+\rho)^2-\rho^2\}^{\frac{1}{2}} = (R^2+2R\rho)^{\frac{1}{2}} + (r^2+2r\rho)^{\frac{1}{2}} \dots (1).$$

The condition that the new circle should touch the common tangent is that the length of this common tangent must be equal the sum of the lengths of the common tangents of the new circle with each of the old circles, *i.e.*,

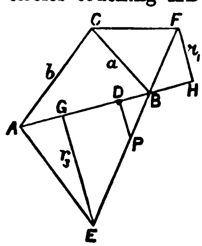
$$\{D^2-(R-r)^2\}^{\frac{1}{2}} = \{(R+\rho)^2-(R-\rho)^2\}^{\frac{1}{2}} + \{(r+\rho)^2-(r-\rho)^2\}^{\frac{1}{2}} = 2(R\rho)^{\frac{1}{2}} + 2(r\rho)^{\frac{1}{2}} \dots\dots (2);$$

we get $\rho = 0$, or $= 4Rr\{R+r+4(Rr)^{\frac{1}{2}}\}/[\{(R+r)+8(Rr)^{\frac{1}{2}}\}^2-4Rr]$;

by substituting in (2), we get the value of D, viz.,

$$D^2 = (R-r)^2 + 4\rho(R^{\frac{1}{2}}+r^{\frac{1}{2}})^2$$
$$= (R-r)^2 + 16(R^{\frac{1}{2}}+r^{\frac{1}{2}})^2 Rr\{R+r+4(Rr)^{\frac{1}{2}}\}/[\{(R+r)+8(Rr)^{\frac{1}{2}}\}^2-4Rr].$$

12436. (R. KNOWLES, B.A.)—On AB, a side of a triangle ABC, AD is taken $= \frac{1}{2}(AB+BC)$; prove that the perpendicular from D on AB bisects the line joining the centres of the escribed circles touching AB and BC.



Solution by R. CHARTRES, Professor MATZ, *and others.*

Since EG and FH are the radii of the escribed circles, therefore

$$BG - BH = c - a = 2DB;$$

therefore $\qquad BE - BF = 2PB,$

or P is the mid-point of EF.

9742. (Professor KALIPADA BASU.)—Four normals are drawn from a point on $x^2-y^2 = a^2e^4$ to the conic $x^2/a^2+y^2/b^2-1 = 0$. If $\alpha+\beta = \frac{1}{2}\pi$, where α and β are the angles made with the axis of x by two of the normals, and θ the angle made by the central radius vector with the same axis, prove that $\tan\theta = \sin 2\alpha$.

Solution by H. J. WOODALL, A.R.C.S.; Professor MOREL; *and others.*

Let PA, PB be the normals from P (on the hyperbola) to points A, B on the ellipse.

Their equations will be $\quad ax\sec\phi - by\operatorname{cosec}\phi = a^2-b^2 \dots\dots (1),$

and $\quad ax\sec\phi' - by\operatorname{cosec}\phi' = a^2-b^2 \dots\dots (2).$

Since P is on the hyperbola, we may put its coordinates

$$x = ae^2 \sec\psi = (a^2-b^2)\sec\psi/a, \quad y = (a^2-b^2)\tan\psi/a \ \ldots\ldots(3, 4).$$

Also $POx = \theta$, whence $\quad \tan\theta = \sin\psi \ \ldots\ldots(5).$

Again $\quad \tan\alpha = a\tan\phi/b = b\cot\phi'/a \ \ldots\ldots(6, 7),$

since $\quad \alpha+\beta = \frac{1}{2}\pi.$

Substituting from (3) to (7) in (1), (2), we get

$$\frac{(a^2-b^2)}{(1-\tan^2\theta)^{\frac{1}{2}}}\cdot\frac{(a^2+b^2\tan^2\alpha)^{\frac{1}{2}}}{a}-\frac{b}{a}\,\frac{(a^2-b^2)\tan\theta}{(1-\tan^2\theta)^{\frac{1}{2}}}\cdot\frac{(a^2+b^2\tan^2\alpha)^{\frac{1}{2}}}{b\tan\alpha} = (a^2-b^2) \ \ldots\ldots(8),$$

$$\frac{(a^2-b^2)}{(1-\tan^2\theta)^{\frac{1}{2}}}\cdot\frac{(b^2+a^2\tan^2\alpha)}{a\tan\alpha}-\frac{b}{a}\,\frac{(a^2-b^2)\tan\theta}{(1-\tan^2\theta)^{\frac{1}{2}}}\cdot\frac{(b^2+a^2\tan^2\alpha)^{\frac{1}{2}}}{b} = (a^2-b^2) \ \ldots\ldots(9),$$

giving $\quad (a^2+b^2\tan^2\alpha)^{\frac{1}{2}}(\tan\alpha-\tan\theta) = a\tan\alpha\,(1-\tan^2\theta)^{\frac{1}{2}} \ \ldots\ldots(8)',$

$$(b^2+a^2\tan^2\alpha)^{\frac{1}{2}}(1-\tan\alpha\tan\theta) = a\tan\alpha\,(1-\tan^2\theta)^{\frac{1}{2}} \ \ldots\ldots(9)'.$$

Eliminate $a : b$, and we get

$$\sec^2\alpha\tan\theta(\sin 2\alpha-\tan\theta)(1-\tan\alpha\tan\theta)^2$$
$$= \tan^4\alpha\sec^2\alpha\tan\theta\,(\sin 2\alpha-\tan\theta)(\tan\alpha-\tan\theta)^2;$$

therefore $\quad \tan\theta = \sin 2\alpha.$

11413. (Professor Barisien.)—On considère une ellipse (E) et une droite (D) perpendiculaire à l'une des diagonales du rectangle des axes. D'un point M de la droite D, on abaisse des normales à l'ellipse. Montrer que le quadrilatère des pieds des normales est un trapèze dont les bases sont parallèles à une diagonale du rectangle des axes. Lorsque le point M se déplace sur la droite D, les côtés non parallèles et les diagonales du trapèze enveloppent une hyperbole équilatère, et le lieu des pôles de ces droites, par rapport à l'ellipse, est aussi une hyperbole équilatère.

Solution by W. J. Greenstreet, M.A.; Professor Basu; *and others.*

Let the ellipse E be referred to the equi-conjugate diameters as axes. Then, its equation being $x^2+y^2-\frac{1}{2}(a^2+b^2) = 0$, that of the hyperbola of Apollonius of (α, β) M will be

$$(x^2-y^2)(a^2-b^2)-\left\{(a^2-b^2)\,\alpha-(a^2+b^2)\,y\right\}x-\left\{(a^2+b^2)\,\alpha-(a^2-b^2)\,y\right\}y = 0 \ \ldots\ldots(1).$$

And as α, β lies on D, $\alpha/\beta = (a^2-b^2)/(a^2+b^2)$, and (1) reduces to

$$(x^2-y^2)(a^2-b^2)^2+4a^2b^2\alpha x = 0,$$

the parabola through the feet of the normals will be

$$2(x^2+y^2)-(a^2+b^2)+\lambda\left\{(x^2-y^2)(a^2-b^2)^2+4a^2b^2\alpha x\right\} = \mu = 0.$$

The terms of the second degree being a perfect square, we have

$$\lambda = \pm 2/c^4 \quad \text{and} \quad \mu = 4x^2 + 8a^2b^2ax/c^4 - (a^2+b^2) = 0,$$

whence $\qquad x_1 + x_2 = -2a^2b^2a/c^4, \quad -x_1x_2 = \frac{1}{4}(a^2+b^2).$

Hence, if ABCD are the feet of the normals, and d one of the diagonals of the rectangle, the quadrilateral is a trapezium, the bases of which are parallel to the other diagonal.

If (x_1, y_1) be the coordinates, A, (x_2, y_2) those of C, we have for the poles of AC the equations

$$2(xx_1 + yy_1) - (a^2+b^2) = 0, \quad 2(x_1^2 + y_1^2) - (a^2+b^2) = 0,$$
$$2(xx_2 + yy_2) - (a^2+b^2) = 0, \quad 2(x_2^2 + y_2^2) - (a^2+b^2) = 0,$$

and $\qquad x_1x_2 = -\frac{1}{4}(a^2+b^2).$

Eliminating y_1 and y_2, we have x_1 and x_2 the roots of

$$4(x^2+y^2)p^2 - 4(a^2+b^2)xp + (a^2+b^2)(a^2+b^2-2y^2) = 0,$$

so that $\quad x_1x_2 = \dfrac{(a^2+b^2)(a^2+b^2-2y^2)}{4(x^2+y^2)} = -\frac{1}{4}(a^2+b^2)$, or $x^2 - y^2 = b^2 - a^2$,

the locus of the poles of the diagonals and the intersecting sides. And the polar reciprocal of $x^2 - y^2 = b^2 - a^2$ with respect to E is an equilateral hyperbola, and is the envelope of the diagonals and intersecting sides.

11520. (Professor NEUBERG.)—Trouver

$$\int \frac{\sin 3x}{\cos 4x}\,dx.$$

Solution by VINCENT J. BOUTON, B.Sc.

$$\int \frac{\sin 3x}{\cos 4x}\,dx = -\int \frac{(4\cos^2 x - 1)\,d(\cos x)}{8\cos^4 x - 8\cos^2 x + 1}.$$

Let $\cos x = z$: then $\displaystyle\int \frac{\sin 3x}{\cos 4x}\,dx = -\int \frac{4z^2-1}{8z^4 - 8z^2 + 1}\,dz.$

Let $\displaystyle\frac{4z^2-1}{8z^4-8z^2+1} = \frac{A}{z^2-\alpha^2} + \frac{B}{z^2-\beta^2}$, where $\alpha^2 = \dfrac{2+\sqrt{2}}{4}$, $\beta^2 = \dfrac{2-\sqrt{2}}{4}$.

Then $\qquad A = \dfrac{2+\sqrt{2}}{8}, \quad B = \dfrac{2-\sqrt{2}}{8};$

therefore $\displaystyle\int \frac{\sin 3x}{\cos 4x}\,dx = -\frac{2+\sqrt{2}}{8}\int \frac{dz}{z^2-\alpha^2} - \frac{2-\sqrt{2}}{8}\int \frac{dz}{z^2-\beta^2}$

$$= -\frac{\sqrt{(2+\sqrt{2})}}{8}\log\frac{\cos x - \alpha}{\cos x + \alpha} - \frac{\sqrt{(2-\sqrt{2})}}{8}\log\frac{\cos x - \beta}{\cos x + \beta}.$$

[Prof. NEUBERG remarks that: "Comme $\cos 4x = 0$ pour $4x = \frac{1}{2}\pi$ ou $\frac{3}{2}\pi$, on voit que $\alpha = \cos\frac{1}{8}\pi$, $\beta = \cos\frac{3}{8}\pi$, ce qui permet de mettre l'intégrale sous une forme plus simple."]

10388. (J. J. Barniville, B.A.)—Prove that the formula for

$$[1^2]2^{-0}+[1^2+2^2]2^{-1}+[1^2+2^2+3^2]2^{-2}+\ldots = 24,$$

$$[1\,.\,1^2]2^{-0}+[2\,.\,1^2+1\,.\,2^2]2^{-1}+[3\,.\,1^2+2\,.\,2^2+1\,.\,3^2]\,2^{-2}+\ldots = 48,$$

$$[1\,.\,1^2]\,2^{-0}+[3\,.\,1^2+1\,.\,2^2]\,2^{-1}+[6\,.\,1^2+3\,.\,2^2+1\,.\,3^2]\,2^{-2}$$
$$+[10\,.\,1^2+6\,.\,2^2+3\,.\,3^2+1\,.\,4^2]\,2^{-3}+\ldots = 96,$$

$$[1\,.\,1^2]\,2^{-0}+[4\,.\,1^2+1\,.\,2^2]\,2^{-1}+[10\,.\,1^2+4\,.\,2^2+1\,.\,3^2]\,2^{-2}$$
$$+[20\,.\,1^2+10\,.\,2^2+4\,.\,3^2+1\,.\,4^2]\,2^{-3}+\ldots = 192.$$

Solution by H. J. Woodall, A.R.C.S.; Professor Chakrivarti; *and others.*

Consider the series $1^2/2+2^2/2^2+3^2/2^3+\ldots+p^2/2^p+\ldots = S$;
multiplying by $(1-\frac{1}{2})^3$,
we get $\frac{1}{2}(1^2)+1/2^2\,(2^2-3)+1/2^3\,(3^2-3\,.\,2^2+3)+$ terms which vanish

$$= \tfrac{1}{2}+\tfrac{1}{4}+0 = \tfrac{3}{4};$$

therefore series $= \frac{3}{4}/(1-\frac{1}{2})^3 = 6 = S.$

The first series $= 2S+S+\frac{1}{2}S+1/2^2\,S+\ldots = 4S = 24,$

second ,, $= 4\,(\frac{1}{2}+2/2^2+3/2^3+\ldots)\,S = 2S/(1-\frac{1}{2})^2 = 48,$

third ,, $= 4\,(\frac{1}{2}+3/2^2+6/2^3+\ldots)\,S = 2S/(1-\frac{1}{2})^3 = 96,$

fourth ,, $= 4\,(\frac{1}{2}+4/2^2+10/2^3+\ldots)\,S = 2S/(1-\frac{1}{2})^4 = 192.$

It will be noticed that the coefficients of the first bracket are the figurate numbers of the respective orders; the terms themselves are the terms of the expansion of $(1-\frac{1}{2})^{-k} = 2^k$.

1185. (W. C. Otter, F.R.A.S.)—Suppose a man has a calf which at the end of three years begins to breed, and afterwards brings forth a female calf every year; and that each calf begins to breed in like manner at the end of three years, bringing forth a cow calf every year; and that these last breed in the same manner, &c.; find the owner's stock at the end of x years.

Solution by Profs. Zerr, Mukhopadhyay, *and others.*

The owner's stock at the end of x years will be the xth term of the series $1+y+2y^2+3y^3+4y^4+6y^5+9y^6+13y^7+\ldots$, when $y=1$.

To find the scale of relation, we have

$$3y^3 = 2py^2+ny+m,\quad 4y^4 = 3py^3+2ny^2+my,$$

$$6y^5 = 4py^4+3ny^3+2my^2;\quad \therefore\ n=0,\ p=y,\ m=y^3.$$

Hence the series is the sum of three geometrical series whose ratios are the roots of the equation $r^3 = pr^2+nr+m$, or $r^3 = r^2y+y^3$.

Let r_1, r_2, r_3 be these roots and a_1, a_2, a_3 the first terms of the series $a_1+a_2+a_3 = 1$, $a_1r_1+a_2r_2+a_3r_3 = 1$, $a_1r_1^2+a_2r_2^2+a_3r_3^2 = 2$, gives a_1, a_2, a_3,

$$a_1\,(1+r_1+r_1^2+\ldots+r_1^{x-1}),\quad a_2\,(1+r_2+r_2^2+\ldots+r_2^{x-1}),$$

$$a_3\,(1+r_3+r_3^2+\ldots+r_3^{x-1}) \text{ are the series;}$$

therefore owner's stock $= a_1r_1^{x-1}+a_2r_2^{x-1}+a_3r_3^{x-1}$.

12496 & 12530. (Rev. T. P. Kirkman, M.A., F.R.S.)—(12496) $U = 0$ is any equation of the mth degree ($m > 2$, odd or even) which has, after the first, n different rational and integral coefficients alternately $+$ and $-$, and which has any finite roots, rational or not, and real or not. $V = 0$ differs from $U = 0$ only by one unit more in the last term, which is L in U, and $L+1$ in V. Desired a demonstration that $V = 0$ has no finite root whatever, or proof, with an example, of the contrary.

(12530) Show that the common belief that $U = x^3 - ax^2 + bx - c = 0$ can be logically deprived of its second term, whatever be the rationals a, b, c, is erroneous; and thence value the opinion that every such $U = 0$ has a root.

Solution by the Proposer.

1. $U = 0$ of the fourth degree below is any true equation known to have four finite roots whose rational parts are not all equal; and $V = 0$ is affirmed, and by hypothesis granted, but not proven, to be also an equation that has roots. From

$$x^4 - Ax^3 + Bx^2 - Cx + L = 0 = U, \quad x^4 - Ax^3 + Bx^2 - Cx + L + 1 = 0 = V,$$

follows, if q be any root of U, and r be any root of V,

$$\frac{q^4 - r^4}{q-r} - A\frac{q^3 - r^3}{q-r} + B\frac{q^2 - r^2}{q-r} - C = \frac{1}{q-r} \text{.................(H)},$$

by forming $U-V$, and then dividing by $q-r$.

2. Let the integer q_1 be a least root of U, that is, a root whose rational part is not greater than that of any other root of U; and let the integer r be *any* root of V.

3. As the divisions in the left member of H are without remainder, that member, being an integer, cannot be equal to the irreducible fraction on the right. We are, like Euclid *passim*, compelled, in order to have in it a possible datum, to write either $q_1 - r = 1$ or $q_1 - r = -1$, *i.e.*, either

$$q_1 = r + 1 \quad \text{or} \quad r = q_1 + 1 \text{........................(1, 2)}.$$

4. Let the first of these be true. Since $q_1 - r$ is rational, the irrational parts of q_1 and of r are equal, and that difference is $q_1' - r' = 1$, where q_1' and r' are the rational parts of q_1 and of r. From this, since r is *any* root of $V = 0$, it follows that $V = 0$ has only one root, whose irrational part is that of q_1, and whose rational part r', is less by unity that the like part q_1' of the root q_1. Our result thus far is

$$r' = q_1' - 1 \text{ ....(3)}.$$

5. Let now $q_2 = q_1 + e$ be a least root but one of U, whose rational part is $q_2' > q_1'$, r being still any root of $V = 0$, and r' its rational part. Our data are now the equation H in q_2 and r, and the equation (1),

$$q_2 (= q_1 + e) = r + 1,$$

where e is rational and finite > 0.

We conclude, as above, in (4), from the difference $q_2 - r = 1$, q_2', q_1, and r' being the rational parts of the roots q_2, q_1, r, that that difference is exactly $q_2' - 1 = r'$, or $q_1' + e - 1 = r'$, which by (3) is $r' + = r'$.

We have thus demonstration that $V = 0$ has no root r whose rational

part r' is a finite number. We have, I mean, proved that, if the equation (1) is true for every q and r. But (2) may be the true relation between q and r, giving for that between q_1' and r' [*vide* (3), q_1 and q_2 being as above], instead of $q_1'-r=1$, and $q_2'-r'=1$, simply $q_1'-r'=-1$, and $q_2'-r'=-1$. Our result from $q_2'+1=r'$, last written, which is $q_1'+e+1=r'$, and from $q_1'+1=r'$, just above written, is again $r'+e=r'$, proving that r' is no finite number, whichever of (1, 2) be true.

It is easily seen that this demonstration would have been got with equal ease, if the equations $U=0$ and $V=0$ had been of the degree $2m$.

But we have not yet overturned the orthodox dogma in all languages that every equation of odd degree has a root.

Let U be multiplied by $x-1=0$. The results of this and of adding $+1$ to the final term are

$$U = x^5-Ax^4+Bx^3-Cx^2+Dx-L=0,$$

$$V = x^5-Ax^4+Bx^3-Cx^2+Dx-L+1=0,$$

whence we get, as before,

$$\frac{q^5-r^5}{q-r}-A\frac{q^4-r^4}{q-r}+B\frac{q^3-r^3}{q-r}-C\frac{q^2-r^2}{q-r}=\frac{1}{q-r}\ \ldots\ldots\ldots\ldots(H').$$

We demonstrate from (H′) as easily as from (H) above, by the very same steps, that such a $V=0$, whether of odd or of even degree, has no finite root whatever. And to reason from U and V of degree $2m$ and $2m+1$ would not make the truth clearer.

By what precedes my object in proposing Quests. 12496 and 12530 is completely gained, and my propositions, along with an assertion in Question 12555, are demonstrated.

It is important to observe that in working out a *reductio ad absurdum*, from clearly given data and granted hypothesis, we are not bound to consider nonsense that may lurk in our deductions when they are read as our own direct propositions. All that we are bound to do is to see that we make no deductions that do not issue of equal logical necessity from the undeniably clear data and from the no less clearly defined hypothesis which we combat. This is a law in honest *reductio ad absurdum*, of which I have known profound philosophers and even high mathematicians in high places to be as unconscious as any learned young lady.

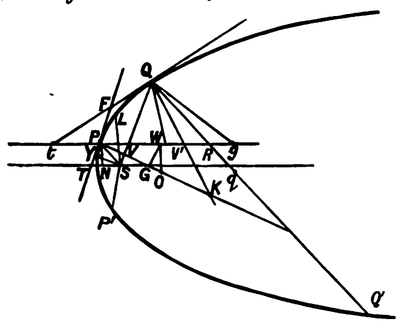
The reader can add richly to the nonsense deduced in § 4.

I am of opinion that, because neither of the equations (1, 2) is a more evident deduction from the data and hypothesis than the other, while one of them of necessity is true, and while each is alike a direct contradiction to the other, we have a right to add them together, and to take impartially the answer which they conspire to give to our demand, What is this r? That answer, without more ado, is here $r=r+2$.

12319. (C. E. Hillyer, M.A.)—FP, FQ are two tangents to a conic, and the circle FPQ meets the diameter through P in g; if QV be the ordinate of Q to this diameter, and QV′ be drawn equally inclined with

QV to the diameter, prove that in the parabola $V'g$ is constant, and in a central conic $V'g$ bears a constant ratio to the abscissa CV. Hence, by making Q move up to R, evaluate the radius of curvature at P, taking the centre of curvature to be the intersection of consecutive normals, and show that the common chord of the conic and the circle of curvature at P is equally inclined with the tangent at P to the major axis.

Solution by the Proposer.



Let QF meet gP in t. Then $QV'g$ and tVQ are similar triangles; therefore

$$\frac{V'g}{QV'} = \frac{VQ}{tV}.$$

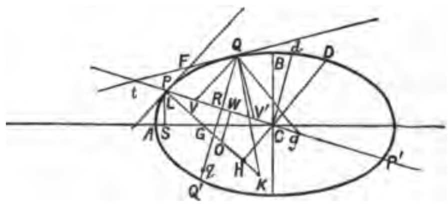
But $QV' = QV$; therefore

$$V'g = \frac{QV^2}{tV}.$$

Now in the parabola

$$QV^2 = 4SP \,.\, PV$$
$$= 2SP \,.\, tV.$$

Hence $V'g = 2SP$, and in a central conic whose centre is C,



$$\frac{QV^2}{CV \,.\, tV} = \frac{CD^2}{CP^2};$$

therefore

$$\frac{V'g}{CV} = \frac{CD^2}{CP^2}.$$

If the normals at P and Q meet in K, K is on the circle $PFQg$, and FgK is a right angle; thus if PW be taken equal to 2SP in the parabola, or equal to CD^2/CP in a central conic, and WO drawn at right angles to PW to meet PK in O, W is the ultimate position of g when Q moves up to P, and O the ultimate position of K, and therefore the centre of curvature, because it is the intersection of consecutive normals.

First taking case of parabola, if SY be drawn perpendicular to PT the tangent at P,

$$WPO = OPS = PSY;$$

therefore

$$\frac{OP}{PW} = \frac{SP}{SY};$$

therefore radius of curvature

$$OP = \frac{SP \,.\, PW}{SY} = \frac{2SP^2}{SY}.$$

Again, since $PW = 2SP = TG$, PTGW is a parallelogram; therefore

WGP is a right angle, and $PO . PG = PW^2 = TG^2$; therefore

$$PO = \frac{TG^2}{PG} = \frac{PG . TG}{NG}\left(\text{since } \frac{TG}{PG} = \frac{PG}{NG}\right) = \frac{PG^3}{NG^2}\left(\text{since } \frac{TG}{NG} = \frac{PG^2}{NG^2}\right);$$

therefore radius of curvature $= \dfrac{PG^3}{SL^2}$.

Also, if any straight line through Q meet the diameter Pg in R, the parabola in Q′ and the circle $PFQg$ in q, and P′ be the extremity of the diameter of parabola bisecting QQ′, we have

$$\frac{QR}{RQ'} = \frac{PV}{PR} \quad \text{and} \quad \frac{QR}{Rq} = \frac{QR^2}{QR . Rq} = \frac{4SP' . PV}{PR . Rg};$$

therefore

$$\frac{Rq}{RQ'} = \frac{Rg}{4SP'}.$$

Now, when Q moves up to P, Rg becomes PW, which equals 2SP, and Rq becomes half the chord of curvature in this direction; therefore chord of curvature in any direction is to corresponding chord of parabola as SP : SP′, *i.e.*, in ratio of focal chord of parabola, parallel to tangent at P and to given direction.

Next, in the case of a central conic, let PO meet CD in H; then OHC is a right angle equal OWC; therefore

$$PO . PH = PW . PC = CD^2,$$

but

$$PG . PH = BC^2; \quad \therefore \frac{PO}{PG} = \frac{CD^2}{BC^2},$$

but

$$\frac{PG}{CD} = \frac{CB}{CA} = \frac{SL}{CB}; \quad \therefore \frac{CD^2}{CB^2} = \frac{PG^2}{SL^2}; \quad \therefore \frac{PO}{PG} = \frac{PG^2}{SL^2};$$

therefore radius of curvature $= \dfrac{PG^3}{SL^2}$,

and, if any chord QQ′ of the conic meet CP in R, and the circle $PFQg$ in q, and Cd be the semi-diameter parallel to QQ′,

$$QR . RQ' = \frac{Cd^2}{CP^2} . PR . RP',$$

and

$$QR . Rq = PR . Rg; \quad \therefore \frac{RQ'}{Rq} = \frac{Cd^2}{CP^2} . \frac{RP'}{Rg},$$

but when Q moves up to P,

$$RP' = 2CP, \quad \text{and} \quad Rg = PW = \frac{CD^2}{CP},$$

and Rq becomes half the chord of curvature in the given direction, thus ultimately

$$\frac{PQ'}{Pq} = \frac{2Cd^2}{CD^2};$$

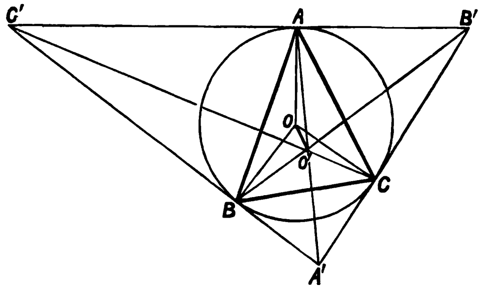
$$\therefore \frac{\text{chord of curvature}}{\text{chord of conic}} = \frac{CD^2}{Cd^2} = \frac{\text{focal chord parallel to tangent}}{\text{focal chord parallel to given direction}}.$$

If QQ′ and FP are equally inclined to the axis, the corresponding focal chords are equal; therefore chord of curvature in this direction is equal to the chord of the conic, *i.e.*, the common chord of conic and circle of curvature at any point is equally inclined to the axis with the tangent at the point.

6566. (The late Professor WOLSTENHOLME, Sc.D.)—The tangents at A, B, C to the circle ABC form a triangle A′B′C′; AA′, BB′, CC′ concur in O′, and O is the centre of the circle ABC; prove that

$$OO' = R\left\{\frac{1-\cos(B-C)\cos(C-A)\cos(A-B)+\cos A\cos B\cos C + \cos 2A\cos 2B\cos 2C}{2(1+\cos A\cos B\cos C)^2}\right\}^{\frac{1}{2}}.$$

Solution by H. J. WOODALL, A.R.C.S.



In the figure we have the following values of sides and angles:—

$$BC = 2R\sin A,\quad CA = 2R\sin B,\quad AB = 2R\sin C;$$
$$AO = BO = CO = R;\quad CAB = A,\quad ABC = B,\quad BCA = C;$$
$$BOC = 2A,\quad COA = 2B,\quad AOB = 2C;$$
$$BA'C = \pi - 2A,\quad CB'A = \pi - 2B,\quad AC'B = \pi - 2C.$$
$$A'BC = A'CB = A,\quad B'CA = B'CB = B,\quad C'AB = C'BA = C;$$
$$BA' = CA' = R\tan A,\quad CB' = AB' = R\tan B,\quad AC' = BC' = R\tan C;$$
$$B'C' = R(\tan B + \tan C) = R\sin A\sec B\sec C,$$
$$C'A' = R\sin B\sec C\sec A,\quad A'B' = R\sin C\sec A\sec B;$$
$$A'O = R\sec A,\quad B'O = R\sec B,\quad C'O = R\sec C;$$

$$AA'^2 = AB'^2 + A'B'^2 - 2AB'.A'B'.\cos AB'A'$$
$$= R^2\sec^2 A(\sin^2 A + 4\cos A\sin B\sin C),$$
$$AA' = R\sec A\left\{\sin^2 A + 4\cos A\sin B\sin C\right\}^{\frac{1}{2}},$$
$$BB' = R\sec B\left\{\sin^2 B + 4\cos B\sin A\sin C\right\}^{\frac{1}{2}},$$
$$CC' = R\sec C\left\{\sin^2 C + 4\cos C\sin A\sin B\right\}^{\frac{1}{2}};$$
$$\frac{AO'}{AB} = \frac{\sin ABO'}{\sin AO'B} = \frac{\sin(C'BO' - C'BA)}{\sin(AA'B + A'BB')},$$

expanding and evaluating which, we get, finally,

$$AO' = AA'\cos A\sin B\sin C/(1+\cos A\cos B\cos C),$$

BO′ and CO′ similarly,

$$\cos O'AO = \cos(\tfrac{1}{2}\pi - O'AB') = \sin O'AB' = A'B' \sin 2B/AA',$$

$$OO'^2 = OA^2 + O'A^2 - 2OA \,.\, O'A \cos OAO',$$

whence $$OO'^2 (1 + \cos A \cos B \cos C)^2/R^2$$

$$= (1 + \cos A \cos B \cos C)^2 - 3 \sin^2 A \sin^2 B \sin^2 C$$

$$= \Sigma \sin^2 B \sin^2 C - 4 \sin^2 A \sin^2 B \sin^2 C$$

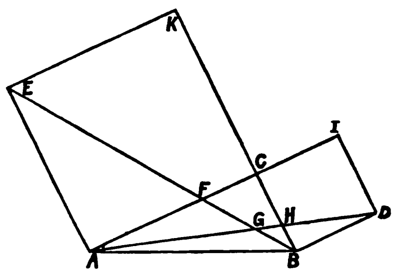
$$= \tfrac{1}{2} \{1 - \cos(B-C) \cos(C-A) \cos(A-B) + \cos A \cos B \cos C + \cos 2A \cos 2B \cos 2C\};$$

therefore $$OO' (1 + \cos A \cos B \cos C)$$

$$= R \left[\tfrac{1}{2} \{1 - \cos(B-C) \cos(C-A) \cos(A-B) + \cos A \cos B \cos C + \cos 2A \cos 2B \cos 2C\}\right]^{\frac{1}{2}}.$$

12463. (M. Brierley.)—Let ABC be a right-angled triangle, and squares ACKE, BCID drawn upon the legs AC, BC. Join A, D, B, E; the lines AD, BE, intersecting in G, form a triangle ABG, and a quadrilateral FCHG, in ABC. Prove that FCHG = ABG.

Solution by T. Savage; Prof. Sanjána; *and others.*



$\Delta EAB = \tfrac{1}{2}$ rect. AK

$= \Delta EAF + \Delta KCF$;

therefore

$\Delta AFB = \Delta KCF$

$=$ (Euc. I. 4) ΔACH;

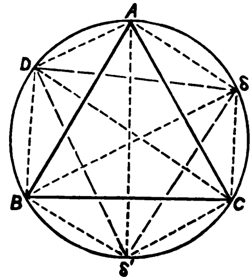
therefore

$\Delta AGB = FCHG$.

12506. (Professor de Wachter.)—Si les perpendiculaires abaissées des sommets d'un triangle ABC sur les côtés correspondants d'un triangle A′B′C′ concourent en un même point D, les perpendiculaires abaissées des sommets de A′B′C′ sur les côtés correspondants de ABC concourent en un même point D′. Dans ce cas on dit que les deux triangles ABC et A′B′C′ sont *orthologiques*, que D est le *métapôle* du couple ABC, A′B′C′, et que D′ est celui du couple A′B′C′, ABC (terminologie de MM. Lemoine et Neuberg). A prouver que deux triangles sont triplement orthologiques, s'ils le sont doublement, et que le terne de métapôles du triangle donné ABC parcourt l'ellipse de Steiner correspondant à ce triangle, si

l'on change le triangle A'B'C' de manière à rester triplement orthologique à ABC.

Solution by Professor Droz-Farny.



Considérons un triangle équilatéral ABC et sa circonférence circonscrite. Par les sommets menons des droites parallèles à une direction quelconque; elles rencontreront la circonférence aux points D, δ, δ'. On démontre aisément que (1) $D\delta\delta'$ est un triangle équilatéral ayant donc avec ABC le barycentre en commun, et (2) que les faisceaux D (ABC), δ (BCA), δ' (CAB) ont leurs rayons homologues parallèles. Faisons sur un plan quelconque une projection orthogonale; on obtient le théorème suivant. Soit E l'ellipse de Steiner circonscrite au triangle ABC, et D un point quelconque de cette dernière que l'on joint aux sommets du triangle; il existe sur l'ellipse deux autres points δ et δ' tels que les faisceaux D (ABC), δ (BCA), δ' (CAB) aient leurs côtés homologues parallèles. Les triangles ABC et $D\delta\delta'$ sont barycentriques.

Il suffit de considérer un triangle A'B'C' dont les côtés sont respectivement perpendiculaires sur DA, DB, DC; ce triangle sera triplement orthologique avec ABC.

5263 & 10905. (Artemas Martin, LL.D.)—**If four pennies be piled up at random on a horizontal plane, what is the probability that the pile will not fall down?**

Note by the Editor.

To Mr. Biddle's criticism of Mr. Heaton's solution of these problems, and to the editorial note appended thereto (Vol. lxii., p. 48), we have received the following replies. A solution of the problem had been given by Mr. Curjel (Vol. lxi., p. 114) virtually agreeing with Mr. Biddle's in method and result.

(1) Mr. Heaton observes that "the Editor is correct in saying that Mr. Heaton's solution proceeds on the assumption that each penny must stand securely on that below it before another is placed above. I happened to know, from solutions of similar problems by the Proposer and others, that that was his intention, and it did not occur to me that it was possible to put any other construction upon the words. I now see that it is. Which is the more obvious construction every reader must judge for himself. Mr. Biddle's understanding of it makes a much simpler problem. I should like to have his criticism of the solution with the understanding that the pile must not fall during the act of piling as well as afterwards."

(2) Dr. Artemas Martin says that Mr. Heaton's view of the

question accords with his own; and in his opinion it is the correct one. The statement 'piled up on a horizontal plane' clearly implies that the pile must be built from the bottom upwards, and the phrase 'at random' does not affect this meaning. How a pile of coins *could* be built from the top downwards on a horizontal plane puzzles Mr. Heaton. Mr. Biddle is for *simultaneous* piling, while Mr. Heaton is for *successive* piling from the bottom upwards, and correctly 'proceeds upon the assumption that each penny must stand securely on the one below it before another is placed above,' and 'at random' does *not* affect this limitation. The pile must surely be stable during the piling as well as after it is completed, and the enunciation clearly warrants that assumption. The complete stability of the pile requires, that (a) the centre of the second coin be on the surface of the first or bottom coin; (b) the centre of the third coin be on the surface of the second, and the common centre of gravity of the third and second coins must be over the surface of the bottom coin; and (c) the centre of the fourth coin must be on the surface of the third, the common centre of gravity of the fourth and third coins over the surface of the second coin, and also the common centre of gravity of the fourth, third, and second coins over the surface of the bottom coin. See solutions of Question 2700, which requires the probability of the stability of a pile of three equal coins, pp. 35, 36 and pp. 111, 112, of Vol. xiii., and solution of Quest. 6298, pp. 43, 44, of Vol. xxxiv. See also the *Mathematical Visitor*, Vol. i., No. 1, pp. 9, 10, and the *Mathematical Magazine*, Vol. ii., No. 6, pp. 100, 101, solution of problem 120. In all these solutions it is assumed (quite correctly too, as Mr. Heaton believes) that the pile stands without being held up during the process of piling, except in the first part of Mr. Woolhouse's solution of Quest. 2700, pp. 35, 36, of Vol. xiii., of the *Reprint*. Dr. Martin adds that, Mr. Biddle's solution is correct according to his view of the problem, but not according to my view of it." He would doubtless say the same of Mr. Curjel's solution.

Having pondered the foregoing observations, Mr. Biddle remarks that "he was himself of the same opinion as Dr. Martin and Mr. Heaton, until he came to consider how impossible it would be to ensure the fixity of the pennies already placed, during the act of superposing another, and yet regard the whole as piled 'at random.' If shifting were allowed, it would not only have to be purely accidental, but within the limits above referred to by Dr. Martin, and this would at once destroy the randomness, besides obscuring the regularity and orderliness of the series of possible arrangements. Mr. Heaton, in his solution, says, 'the centre of each penny is liable to *fall* anywhere,' &c. This is a true expression of randomness, and it gives a very different conception from the exceeding gingerliness with which we are now told the several pennies must be piled up. In the opinions of Messrs. Curjel, Biddle, and Woolhouse, it is best to allow the piling to be carried out anyhow, that is, without restrictions of any kind; and then, it matters not whether we consider the arrangements as made from below upwards (the actual way) or from above downwards (the imaginary)."

Mr. Curjel remarks that the solutions differ because the Question is ambiguous. He thinks his own interpretation and Mr. Biddle's is by far *the more reasonable;* Mr. Heaton and Dr. Martin think otherwise. And thus the matter must remain, so far as we are concerned.]

12487. (H. W. SEGAR, B.A.)—Let the numerical series $u_1, u_2, \ldots$ be recurring. If the scale be $u_r = pu_{r-1} + qu_{r-2}$, then, if $q = 1$, all the points having two successive terms for coordinates lie on a conic. If the scale be $u_r = pu_{r-1} + qu_{r-2} + ru_{r-3}$, then, if $p^2 - pr - q - 1 = 0$, all points having three successive terms for coordinates lie on a quadric.

Solution by Professors SCHOUTE, LAMPE, *and others.*

If $(\kappa + lz)/(\gamma - \beta z + \alpha z^2)$ represents the generating fraction of the recuring series $\sum\limits_0^\infty u_n z^n$, the transformation of this generating fraction into the two simple fractions $\lambda(1 - a_1 z) + \mu(1 - a_2 z)$ gives the relation $u_n = \lambda a_1^{\,n} + \mu a_2^{\,n}$. So, according to the conditions of the problem, we have to eliminate n, a_1, a_2 from the four equations

$$x = \lambda a_1^{\,n} + \mu a_2^{\,n}, \quad y = \lambda a_1^{n+1} + \mu a_2^{n+1}, \quad a_1 + a_2 = \beta/\alpha, \quad a_1 a_2 = \gamma/\alpha.$$

We find $\qquad a_1 x - y = \mu a_2^{\,n}(a_1 - a_2), \quad a_2 x - y = \lambda a_1^{\,n}(a_2 - a_1).$

These equations show that the relation between x and y only will be algebraical in the cases $a_1 a_2 = \pm 1$, *i.e.*, $\gamma = \pm \alpha$.

1. The case $\gamma = -\alpha$ is that of the problem. Then we have to distinguish between n even and n odd, as we find

$$(a_1 x - y)(a_2 x - y) = (-1)^{n+1} \lambda\mu (a_1 - a_2)^2,$$

or, with the notation of the problem,

$$x^2 + pxy - y^2 = \mp \lambda\mu (p^2 + 4).$$

For the locus of the point $(x = u_{2n},\ y = u_{2n+1})$ the sign $-$, for that of the point $(x = u_{2n+1},\ y = u_{2n+2})$ the sign $+$, holds. So the locus consists of two conjugated rectangular hyperbolæ.

2. In the case $\gamma = \alpha$, we find only one locus, viz.,

$$x^2 - pxy + y^2 = -\lambda\mu (p^2 + 4).$$

2. In the second example, we have generally

$$x = \lambda a_1^{\,n} + \mu a_2^{\,n} + \nu a_3^{\,n}, \quad y = \lambda a_1^{n+1} + \mu a_2^{n+1} + \nu a_3^{n+1}, \quad z = \lambda a_1^{n+2} + \mu a_2^{n+2} + \nu a_3^{n+2},$$

where a_1, a_2, a_3 are the roots of $\delta - \gamma z + \beta z^2 - \alpha z^3 = 0$. Here the elimination of n conducts to a curve in space, not to a surface. In general the curve is a transcendental one, and not a single algebraical surface passes through it. In particular cases an algebraical surface containing the curve may be assigned. So, for instance, if we are given $a_1 a_2 a_3 = \pm 1$, *i.e.*, $r = \pm 1$, then we find the cubic surface

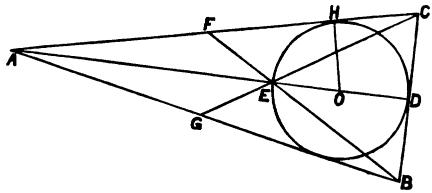
$$\begin{vmatrix} x & 1 & 1 \\ y & a_2 & a_3 \\ z & a_2^2 & a_3^2 \end{vmatrix} \begin{vmatrix} 1 & x & 1 \\ a_1 & y & a_3 \\ a_1^2 & z & a_3^2 \end{vmatrix} \begin{vmatrix} 1 & 1 & x \\ a_1 & a_2 & y \\ a_1^2 & a_2^2 & z \end{vmatrix} = \pm \lambda\mu\nu \begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{vmatrix}^3,$$

the equation of which is easily cleared of a_1, a_2, a_3, &c.

2916. (I. H. TURRELL.)—To construct a triangle geometrically, having given the vertical angle and the radius of the inscribed circle, when the centre of gravity is on the circumference of the inscribed circle.

Solution by MORGAN BRIERLEY.

Let ACB = given vertical angle, and OH the radius of the inscribed circle. Through O, the centre, draw the line DOEA, meeting CA in A, so that AD = 3ED = 6OH. From A draw the tangent AGB, meeting CB in B; ABC is the required triangle. E is obviously the centre of gravity of the triangle, and is in the circumference of the inscribed circle.



The triangle is necessarily isosceles, AC = AB, respectively bisected by the median lines BEF, CEG, and CB by AED.

6478. (Professor EVANS, M.A.)—A tetrahedron ABCD is cut by a plane that passes through A′, C′, the middle points of two opposite edges; prove (1) that A′C′ bisects the quadrilateral section A′B′C′D′; and (2) that the quadrilateral A′B′C′D′ divides the tetrahedron into two equivalent solids.

Solution by H. J. WOODALL, A.R.C.S.; Prof. MOREL; *and others.*

Consider A′B′C′D′ as the plane of the paper. Let A_1, B_1, C_1, D_1 be the projections of the points A, B, C, D on the plane. Then, because

$$AC' = C'D,$$

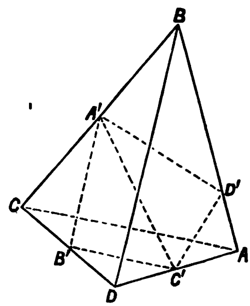
we have $AA_1 = -DD_1 = x$ (say) ...(1);

also $BA' = A'C$

gives $BB_1 = -CC_1 = y$ (say) ...(2).

Therefore

$$BD' : D'A = y : x = CB' : B'D \text{...(3, 4)}.$$



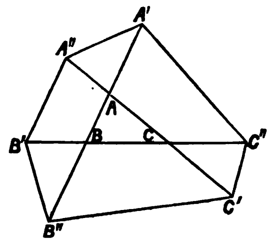
From (1), (2) we find that the perpendicular distances from A′C′ of B and C and also of A and D are equal. This result with (3) and (4) proves that D′ and B′ are equally distant from A′C′; hence the triangles A′C′D′ and A′C′B′ are equivalent.

2. When a plane is drawn parallel to BC and AD to cut the tetrahedron in HKLM, being parallel to BC and AD, it will cut BA and CD proportionally, and therefore in points H, L, which are coplanar with the line A′C′, and therefore H, L are equidistant from the plane A′B′C′D′. K and M are also equidistant from that plane. Hence it follows that each such section is bisected by that plane, and thus the sum total of all such sections.

12422. (Professor SANJÁNA, M.A. Suggested by Quest. 12027.)—The sides AB, AC of a triangle are produced to B″, C′, so that BB″ = CC′ = a; the sides BC, BA to C″, A′, so that CC″ = AA′ = b; and the sides CA, CB to A″, B′, so that AA″ = BB′ = c. Prove that, if α, β, γ stand for sin A, sin B, sin C, the area of A′A″B′B″C′C″ is

$$2R^2\left\{\alpha(\alpha+\beta)(\alpha+\gamma)+\beta(\beta+\gamma)(\beta+\alpha)+\gamma(\gamma+\alpha)(\gamma+\beta)+\alpha\beta\gamma\right\}.$$

Solution by H. W. CURJEL, M.A.; *the* PROPOSER; *and others.*



Each of the triangles AA′A″, BB′B″, CC′C″, is equal to $\triangle$ABC; therefore area of A′A″B′B″C′C″

$$= A''B'C + A'BC'' + AC'B'' + ABC$$

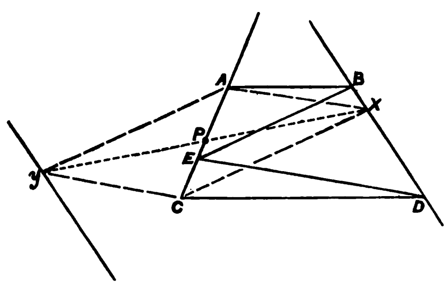
$$= \tfrac{1}{2}\Sigma(a+b)(a+c)\sin A + \triangle ABC$$

$$= 2R^2\left\{\Sigma\alpha(\alpha+\beta)(\alpha+\gamma)+\alpha\beta\gamma\right\}.$$

12519. (Professor BERNÈS.)—Un système de deux droites parallèles AB, CD est coupé par deux sécantes AC, BD. On joint B et D à un point quelconque E de AC. Si par A et C on mène des parallèles respectivement à ED, EB, ces parallèles se coupent sur BD; si par A et C on mène des parallèles à EB, ED, quel est le lieu de leur rencontre?

Solution by Professors DROZ-FARNY, MUKHOPADHYAY, *and others.*

Lorsque le point E se meut sur AC, les rayons DE et BE décrivent



deux faisceaux perspectifs et par conséquent les rayons Ax et Cx qui leur sont respectivement parallèles décrivent deux faisceaux projectifs. Lorsque E coïncide avec le point infini de AC les rayons BE et DE sont parallèles à AC et Ax ainsi que Cx coïncident avec AC. Les faisceaux Ax et Cx sont donc perspectifs. Pour les positions A ou C de E, x coïn-

cide avec D ou avec B ; le lieu des points d'intersection des rayons homologues de ces deux faisceaux est donc la droite BD.

Pour une position donnée de E menons Ay parallèle à BE et Cy parallèle à DE ; la figure yAxC est un parallélogramme, donc xy coupe AC en son point milieu P.

Le point y décrit donc la symétrique de BD par rapport au point P, c'est-à-dire une droite parallèle à BD.

[Mr. Dobbs adds the following extension of the theorem :—A, B, C, D are fixed points. Any point E is taken in CA, and BE, DE intersect a fixed straight line drawn through the intersection of DC and BA in β, δ. Then the locus of the intersection of Aδ, Cβ is BD, and the locus of the intersection of Aβ, Cδ is a straight line passing through the intersection of BD and $\beta\delta$. The whole thing is intimately connected with Pascal's theorem.]

12443. (Professor Lampe, LL.D.)—The initial velocity c of a heavy body being supposed to be given, prove that the length of its parabolic path is a maximum for the elevation α obtained from the equation

$$1-\sin\alpha\log\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\alpha\right)=0.$$

Solution by Professors Sanjána, M.A., Bhattacharya, *and others.*

The coordinates of the point of projection with regard to the vertex are

$$x=c^2\sin^2\alpha/g,\quad y=c^2\sin\alpha\cos\alpha/g\ ;$$

also the *latus rectum* of the trajectory

$$2m=2c^2\cos^2\alpha/g.$$

Now, in the parabola

$$s=\frac{y\,(y^2+m^2)^{\frac{1}{2}}}{2m}+\frac{m}{2}\log\left(\frac{y+(y^2+m^2)^{\frac{1}{2}}}{m}\right).$$

Thus here $s=\dfrac{c^2}{2g}\left\{\sin\alpha+\cos^2\alpha\log\left(\dfrac{1+\sin\alpha}{\cos\alpha}\right)\right\}=\dfrac{c^2}{2g}\,\phi(\alpha)$;

so that $\qquad \phi'(\alpha)=2\cos\alpha-2\sin\alpha\cos\alpha\log\left(\dfrac{1+\sin\alpha}{\cos\alpha}\right)=0,$

for a maximum. This gives

$$1-\sin\alpha\log\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\alpha\right)=0.$$

And $s=\dfrac{c^2}{2g\sin\alpha}$; therefore $2s=\operatorname{cosec}56^\circ\,27'\,57''\,c^2/g$. The vertical rectilinear trajectory, corresponding to $\alpha=90^\circ$, is of course a minimum solution of our problem.

4211. (Elizabeth Blackwood.)—A point is taken at random in the surface of a circle, and a random line drawn through it; two other points are then taken at random in its surface : find the chance that they are on opposite sides of the line.

Solution by H. J. WOODALL, A.R.C.S.

Let O be centre of the circle, radius a, P be the point, OP $= x$. Let APA′ be one position of the random line, and let A′PO $= \theta$. Then

$$\text{area AKA}' = \tfrac{1}{2}a^2\left\{\pi - 2\arcsin(x\sin\theta/a) - 2x/a\sin\theta\cos\left[\arcsin(x\sin\theta/a)\right]\right\} = \text{D}.$$

(1) For fixed position of P and fixed position of APA′, probability

$$= 2\text{D}(\pi a^2 - \text{D})/(\pi a^2)^2;$$

(2) For fixed position of P and variable chord, probability

$$= 2\int_0^{\frac{1}{2}\pi}\text{D}(\pi a^2 - \text{D})\,d\theta \Big/ \int_0^{\frac{1}{2}\pi}\pi^2 a^4\,d\theta.$$

If x and θ both vary, then probability

$$= \left\{\int_0^a 2\pi x\,.\,2\int_0^{\frac{1}{2}\pi}\text{D}(\pi a^2 - \text{D})\,d\theta\,.\,dx\right\} \Big/ \left\{\pi^2 a^4\int_0^a 2\pi x\,dx\int_0^{\frac{1}{2}\pi}d\theta\right\}$$

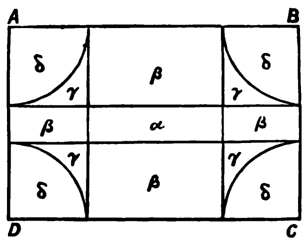
$$= 8\int_0^a x\int_0^{\frac{1}{2}\pi}\text{D}(\pi a^2 - \text{D})\,d\theta\,.\,dx/\pi^3 a^6.$$

6057. (A. MARTIN, LL.D.)—A heavy straight rod is thrown at random on a rectangular table; find the chance (1) that the rod lies wholly *on* the table, (2) that *one* end projects over the edge of the table, and (3) that *both* ends project over the edge of the table.

Solution by H. FORTEY.

(1) In this solution it is assumed that the rod is not longer than the breadth of the table.

(2) Let ABCD be the table of length $2a$ and breadth $2b$. Let $2c$ be the length of the rod, and $a > b > c$. With the corners of the table as centres and c as radius describe four quadrants and draw the other lines in the diagram. Then the area of the table is $4ab$, and it is made up of



1 space	marked	α	of which area		$= 4(a-c)(b-c)$,
4 spaces	,,	β	aggregate	,,	$= 4(a+b-2c)c$,
4	,,	,,	γ	,, ,,	$= (4-\pi)c^2$,
4	,,	,,	δ	,, ,,	$= \pi c^2$.

(3) Call the centre of the rod G. Then G is on the table and all positions are equally likely. Therefore chances that G lies on spaces marked α, β, γ, δ respectively are

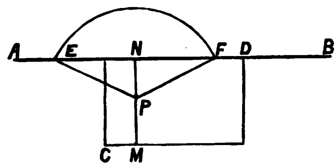
$$\frac{(a-c)(b-c)}{ab},\quad \frac{(a+b-2c)c}{ab},\quad \frac{(4-\pi)c^2}{4ab},\quad \frac{\pi c^2}{4ab}.$$

(4) Let μ be the chance that G lies on a space marked μ, and that n ends project beyond the table (where, of course, $n = 0, 1,$ or 2), and let P_n represent the same chance for the whole table. Then, evidently, $\alpha_1, \alpha_2, \beta_2, \gamma_2, \delta_0$ are all $= 0$, and

$$P_0 = \alpha_0 + \beta_0 + \gamma_0, \quad P_1 = \beta_1 + \gamma_1 + \delta_1, \quad P_2 = \delta_2.$$

(5) Clearly,

$$\alpha_0 = (a-c)(b-c)/ab.$$



(6) Next suppose G lies on a β space. Let AB be an edge of the table, CD a β space, MN perpendicular to AB; then

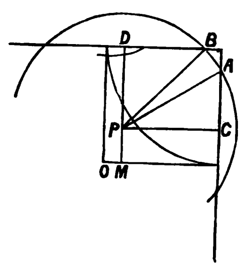
$$MN = c.$$

With any point P in MN as centre and c as radius, describe an arc cutting AB in E and F, and let $PM = y$ and $\angle EPN = \phi$. Then, if G be placed on an element dy of the line MN at P, and the rod be turned through an angle π, the whole angular space about P will be passed over by one half or the other of the rod, and it will project beyond AB through an angle 2ϕ. Therefore the number of projecting positions at $P = 2\phi dy$. But $c - y = c\cos\phi$; therefore $dy = c\sin\phi\, d\phi$. Therefore number of projecting positions at $P = 2c\phi\sin\phi\, d\phi$, and integrating from $\phi = 0$ to $\phi = \frac{1}{2}\pi$, number of projecting positions of G on MN

$$= 2c\int_0^{\frac{1}{2}\pi} \phi\sin\phi\, d\phi = 2c.$$

But whole number of positions in $MN = c\pi$. Therefore chance of projection if G is on $MN = 2/\pi$. And this is true of every line in the area parallel to MN, and the area is made up of such lines. Therefore, if G lies in a β area, the chance of projecting $= 2/\pi$. But chance of G being in a β area $= \dfrac{(a+b-2c)\,c}{ab}$; therefore

$$\beta_1 = \frac{2\,(a+b-2c)\,c}{\pi ab}, \quad \beta_0 = \frac{(\pi-2)(a+b-2c)\,c}{\pi ab}.$$



(7) Next suppose G is on a γ space. Let P be a point in a γ space, and

$$OM = x, \quad PM = y,$$

with P as centre and radius c describe a circle cutting the edges of the table at A, B, and two other points, and let

$$\angle APC = \theta, \quad \angle BPD = \phi.$$

Then, if G be placed on an element $dx\,dy$ P, and the rod turned through an angle π, the number of projecting positions is

$$2\,(\theta+\phi)\,dx\,dy.$$

And $\quad c - x = c\cos\theta, \quad c - y = c\cos\phi$;

therefore $dx = c\sin\theta\, d\theta$, $dy = c\sin\phi\, d\phi$; therefore number of projecting positions $= 2c^2(\theta+\phi)\sin\theta\sin\phi\, d\theta\, d\phi$, and integrating from $\phi = 0$ to

$\phi = \frac{1}{2}\pi - \theta$, and then from $\theta = 0$ to $\theta = \frac{1}{2}\pi$, we have, for number of projecting positions in γ,

$$2c^2 \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi-\theta} (\theta+\phi) \sin\theta \sin\phi \, d\phi \, d\theta = \tfrac{1}{4}c^2(12-\pi^2).$$

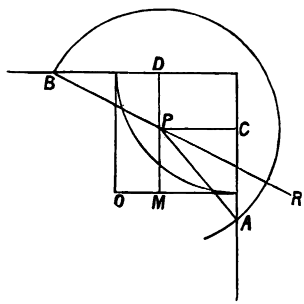
But whole number of positions $=$ area $. \pi = \frac{1}{4}c^2 (4\pi - \pi^2)$; therefore chance of projecting if G is in a γ space $= \dfrac{12-\pi^2}{4\pi-\pi^2}$. But chance that G is in γ space $= \dfrac{(4-\pi)c^2}{4ab}$; therefore

$$\gamma_1 = \frac{(12-\pi^2)c^2}{4\pi ab}, \quad \gamma_0 = \frac{(\pi-3)c^2}{\pi ab}.$$

(8) Finally, let G be on a δ space. Wherever G lies on this space *one* end of the rod *must* project over the edge. Let us therefore determine the chance that *both* ends project.

As before, let

$$OM = x, \quad PM = y.$$



The circle with centre P and radius c will now cut the edges of the table in only one point each, A, B. Let $\angle APC = \theta$, $BPD = \phi$. Produce BP to R. Then, if G is placed on an element $dx\,dy$ at P, both ends of the rod project beyond the table when the rod lies in the angle

$$APR = \theta + \phi - \tfrac{1}{2}\pi.$$

Therefore number of positions of double projection

$$= (\theta+\phi-\tfrac{1}{2}\pi)\,dx\,dy = c^2(\theta+\phi-\tfrac{1}{2}\pi)\sin\theta\sin\phi\,d\theta\,d\phi,$$

and integrating from $\phi = \frac{1}{2}\pi - \theta$ to $\phi = \frac{1}{2}\pi$, and then from $\theta = 0$ to $\theta = \frac{1}{2}\pi$, number of doubly projecting positions in δ

$$= c^2 \int_0^{\frac{1}{2}\pi} \int_{\frac{1}{2}\pi-\theta}^{\frac{1}{2}\pi} (\theta+\phi-\tfrac{1}{2}\pi)\sin\theta\sin\phi\,d\phi\,d\theta = \tfrac{1}{2}c^2.$$

But whole number of positions in $\delta =$ area $. \pi = \frac{1}{4}\pi^2 c^2$. Therefore, if G is in δ, chance of both ends projecting $= 2/\pi^2$; therefore

$$\delta_2 = \frac{\pi c^2}{4\pi ab}\,\frac{2}{\pi^2} = \frac{c^2}{2\pi ab}, \quad \delta_1 = \frac{(\pi^2-2)c^2}{4\pi ab}.$$

(9) Therefore $\quad P_0 = \alpha_0 + \beta_0 + \gamma_0 = \dfrac{\pi ab - 2(a+b)c + c^2}{\pi ab}$,

$$P_1 = \beta_1 + \gamma_1 + \delta_1 = \frac{4(a+b)c - 3c^2}{2\pi ab}, \quad P_2 = \delta_2 = \frac{c^2}{2\pi ab}.$$

(10) In the particular case when $c = b = a$, we have

$$P_0 = \frac{\pi-3}{\pi}, \quad P_1 = \frac{5}{2\pi}, \quad P_2 = \frac{1}{2\pi}.$$

12395. (Professor GALASSI.)—Montrer que l'équation

$$x^3 - y^3 = xy^2(x-2),$$

est impossible en nombres entiers ou fractionnaires.

Solution by T. SAVAGE, Professor BHATTACHARYA, *and others.*

Denoting y by vx, we have $x^2(1-v^3) = x^3v^2(x-2)$, that is, $v(x-1) = \pm 1$. Therefore $y = \pm x/(x-1)$; hence, when x is integral, y is fractional; and *vice versâ.*

9334. (Professor ORCHARD, B.Sc., M.A.)—Evaluate, when $x = 1$,

$$e^{(5^{1-x}-10^{1-x})(x+x^2+x^3+\ldots+x^{x-1})/(\log x+1-x)}.$$

Solution by H. J. WOODALL, A.R.C.S.; Prof. CHAKRIVARTI; *and others.*

Given expression

$$= \text{Lt}\exp\left\{(5^{1-x}-10^{1-x})(x+x^2+\ldots+x^{x-1})/(\log x+1-x)\right\}$$
$$= \exp\left[\text{Lt}\left\{(5^{1-x}-10^{1-x})(x-x^x)/(1-x)(\log x+1-x)\right\}\right]$$
$$= \exp\left[\left\{\text{Lt}(5^{1-x}-10^{1-x})/(1-x)\right\}\left\{\text{Lt}(x-x^x)/(\log x+1-x)\right\}\right]$$
$$= \exp\left[\left\{\text{Lt}(-5^{1-x}\log_e 5+10^{1-x}\log_e 10)/(-1)\right\}\right.$$
$$\left.\times\left\{\text{Lt}(1-x_x-x^x\log_e x)/(1/x-1)\right\}\right]$$
$$= \exp\left[(\log_e 5-\log_e 10)\left\{\text{Lt}\left[0-x^x(1+\log_e x)^2-x^{x-1}\right]/(-1/x^2)\right\}\right]$$
$$= \exp\left[\log_e\tfrac{1}{2}\left\{(0-1-1)/(-1)\right\}\right] = \exp\left[2\log_e\tfrac{1}{2}\right].$$

9936. (Professor NEUBERG.)—Une droite de longueur donnée a se meut en s'appuyant par ses extrémités sur deux droites rectangulaires OX, OY. Quelle est la probabilité que l'aire comprise entre cette droite, OX et OY soit moindre qu'un carré donné q^2?

Solution by H. J. WOODALL, A.R.C.S.

If OX take OA $= a$, let one end of the given line be placed at random at a point P in OA, where OP $= x$, then if Q be the other end (in OY),

$$\text{OQ} = (a^2-x^2)^{\frac{1}{2}};$$

therefore $\quad \text{area} = \tfrac{1}{2}x(a^2-x^2)^{\frac{1}{2}} = q^2;$

therefore $\quad x^2 = \tfrac{1}{2}a^2 \pm (\tfrac{1}{4}a^4-4q^4)^{\frac{1}{2}} = \tfrac{1}{2}\left\{(\tfrac{1}{2}a^2+2q^2)^{\frac{1}{2}} \pm (\tfrac{1}{2}a^2-2q^2)^{\frac{1}{2}}\right\}^2;$

therefore $\quad x = \tfrac{1}{2}(a^2+4q^2)^{\frac{1}{2}} \pm \tfrac{1}{2}(a^2-4q^2)^{\frac{1}{2}} = \text{OP}_1, \text{OP}_2.$

Required probability $= (\text{OA}-\text{P}_1\text{P}_2)/\text{OA}$

$$= \left\{\text{OA}-(\text{OP}_1-\text{OP}_2)\right\}/\text{OA} = \left\{a-(a^2-4q^2)^{\frac{1}{2}}\right\}/a = 1-(1-4q^2/a^2)^{\frac{1}{2}}.$$

APPENDIX.

UNSOLVED QUESTIONS.

3454. (Professor Sylvester.)—Prove that the chance of a group of points satisfying any prescribed condition not explicitly involving reference to linear magnitude remains the same, whether all the points of the group are at liberty to be placed with an equal degree of probability anywhere within a given plane contour; or if this liberty holds only of all of them but one, the remaining one being combined to the contour itself, provided that the probability of the last-named point being found on any arc PQ of the contour be made proportional to the time of describing PQ about a fixed centre of force arbitrarily assumed anywhere within or upon the contour.

3455. (Professor Cayley, F.R.S.)—Indicate in what manner the Lagrangian equations of motion

$$\frac{d}{dt}\frac{d\mathrm{T}}{d\xi'}-\frac{d\mathrm{T}}{d\xi}=\frac{d\mathrm{V}}{d\xi},\ \&c.,$$

lead to the equations $\mathrm{A}\dfrac{dp}{dt}+(\mathrm{C}-\mathrm{B})\,qr=0$, &c.,

for the motion of a solid body about a fixed point.

3456. (Professor Sir R. E. Ball, F.R.S.)—Describe a mechanical arrangement by which a rigid body may be enabled to move freely in every direction about a fixed point either external or internal.

3457. (Rev. T. P. Kirkman, M.A., F.R.S.)—(1) P is a random summit of a random 8-edron. What is the chance that the two other summits, Q, R, on the solid are such that a triangular section PQR of it can be made? (2) F is a random edge of a random 8-edron. What is the chance that the solid has two other edges, such that a triangular section lies through the centre of the three?

3458. (Professor Wolstenholme, M.A.)—If a rod be marked at random in three points, the chance that n times the sum of the squares on the four parts into which the line is divided shall be less than the square on the whole line is

$$\frac{\pi}{2}\left(\frac{4-n}{n}\right)^{\frac{3}{2}}\ \text{ or }\ \frac{\pi}{6\sqrt{3}}\left\{\frac{36-10n}{n}-\left(\frac{3(4-n)}{n}\right)^{\frac{3}{2}}\right\};$$

according as n lies between 3 and 4 or 2 and 3. Obtain the corresponding formula when n lies between 1 and 2.

3463. (Professor Hudson, M.A.)—A solid of resolution possesses this property: A portion being cut off by a plane perpendicular to its axis and immersed vertex downwards in fluid, and then displaced through a

small angle, the moment tending to restore equilibrium is independent of the amount cut off. Show that, if $y = f(x)$ be the generating curve, to determine f we have

$$[f(x)]^2 = e\left\{1+[f'(x)]^2+f(x)f''(x)\right\}\left\{f[x+f(x)f'(x)]\right\},$$

C being the density of the solid compared with the fluid.

3464. (H. MacColl, B.A.)—A certain mathematician solved a question in probability, and obtained q as his expression for the required chance. Not feeling satisfied, however, that this result was correct, he applied the test of experiment, and found that the event in question happened m times in n trials. Supposing P to be a fair estimate of the probability previous to the experiment that the result q was correct, what was the probability after the experiment?

3465. (Editor.)—A point is taken at random on each of the faces of a regular tetrahedron; find the probability that the tetrahedron of which these four points are the corners will be less than one-half of the given one. Also find the average of all such inscribed tetrahedra.

3466. (Professor Genese, B.A.)—A triangle is given; another is formed by joining the points of contact of its inscribed circle; another is formed similarly from this last; and so on. What is the limiting position of the centre of the inscribed circles?

3468. (R. Tucker, M.A.)—A point P is taken upon a circumference; given that one focus of a series of ellipses, which have this circle for the circle of curvature at P, lies upon the circumference; prove that (1) the other focus lies on a circle touching the given one at P, (2) the areas of the ellipses vary as the cubes of the diameters parallel to the common tangent, (3) the envelope of the major axes is a quartic curve posing through P.

3470. (Rev. A. F. Torry, M.A.)—A polar curve (α) passes through the origin; show that its first pedal (β), and (γ) the locus of the extremities of its polar subtangents, cut it there at right angles; and that, if the origin is not a singular point on (α), there are cusps there on (β) and (γ). Show also that the tangents to (α) at its points of inflection touch (γ) and pass through cusps on (β): and that to cusps on (α) correspond points on (γ) the tangents at which pass through the pole, and points on (β) the circles of curvature at which pass through the pole and the cusps on (α). And generally that, according as (α) is convex or concave to the origin, the circle of curvature of (β) lies wholly within or wholly without the circle whose diameter is the radius vector of (α), and the tangent to (γ) cuts this radius or the radius produced.

3471. (Rev. W. A. Whitworth, M.A.)—Find the greatest acute angle at which an ellipse of given eccentricity less than $\left(\dfrac{2}{1+\sqrt{2}}\right)^{\frac{1}{2}}$ can be cut by a circle of curvature. Explain the case of an ellipse of greater eccentricity.

3472. (J. Griffiths, M.A.)—Find the condition that the two conics

$$x^2+y^2+z^2-(l_1x+m_1y+n_1z)^2 = 0, \quad x^2+y^2+z^2-(l_2x+m_2y+n_2z)^2 = 0,$$

shall intersect at an angle θ.

3473. (Artemas Martin, LL.D.)—If a ball be shot into a side of a cube, what is the chance that it will go through the opposite side?

3476. (Rev. C. Taylor, D.D.)—Find an expression for the area of an ellipse which touches one, and has its foci on the other of two fixed confocal ellipses.

3480. (Professor Sylvester.)—Let there be an indefinite number of variables $x, y, z \dots u, v, w \dots$. Let D_i be the number of solutions in positive integers of the equation

$$(x+x')+2(y+y')+\dots+(i-1)(u+u')+iv+(i+1)w+\dots=n-im-\tfrac{1}{2}(i^2-i).$$

Prove that $D_1-D_2+D_3-D_4+D_5\dots$ is the number of solutions in positive integers of the simultaneous system of equations

$$x+y+z+t+\dots=m,\quad x+2y+3z+4t+\dots=n.$$

3485. (J. W. L. Glaisher, B.A.)—Prove that

$$\left(8\int_{p^2}^{\infty}p\,dp\right)^{2i}e^{-2pq}\cos 2pq=q\left(-\frac{d}{q\,dq}\right)^{2i}\frac{e^{-2pq}\cos 2pq}{q},$$

and $$\left(8\int_{p^2}p\,dp\right)^{2i}e^{-2pq}\sin 2pq=q\left(-\frac{d}{q\,dq}\right)^{2i}\frac{e^{-2pq}\sin 2pq}{q}.$$

3487. (J. Griffiths, M.A.)—If the invariants of two conics represented by the equations

$$x^2+y^2+z^2=(lx+my+nz)^2,\quad x^2+y^2+z^2=(l'x+m'y+n'z)^2$$

are connected by the relation

$$\frac{1-(ll'+mm'+nn')}{(1-l^2-m^2-n^2)^{\frac{1}{2}}(1-l'^2-m'^2-n'^2)^{\frac{1}{2}}}=1,$$

it is shown in Dr. Salmon's "Conics," 5th ed., that the conics *touch* each other. What is the geometric meaning of the more general relation

$$\frac{1-(ll'+mm'+nn')}{(1-l^2-m^2-n^2)^{\frac{1}{2}}(1-l'^2-m'^2-n'^2)^{\frac{1}{2}}}=\cos\theta\ ?$$

3489. (Professor Sir R. E. Ball, F.R.S.) $\alpha=0,\ \beta=0,\ \gamma=0,\ \delta=0$ are four lines in a plane; what is the mechanical meaning of the identical formula $$A\alpha+B\beta+C\gamma+D\delta=0,$$ in which A, B, C, D are the areas of the triangles formed by each set of three lines.

3492. (S. Watson.)—Through the angle A of a given triangle ABC, two lines are drawn at random, dividing the triangle into three parts. Also three points are taken at random within the same triangle. Find the respective chances, (1) of all the points lying in some one of the three parts; (2) that two points shall lie in some one of the three parts, one in another, and none in the third; and (3) that one point shall lie in each part.

3503. (Artemas Martin, LL.D.)—A sphere, radius r, and a candle are placed at random on a round table, radius R, the height of the candle being equal to the radius of the sphere. Required the average of the illuminated portion of the surface of the sphere.

3506. (Professor Sylvester.)—(1) By aid of the theorem in Quest. 3480, or otherwise, prove the following theorem:—Let $x, y, z, t, \ldots$ be any system of positive integer values (zeros included) which satisfy the equation (with an unlimited number of variables) $x+2y+3z+4t+\ldots=n$; and call $1-x+xy-xyz+xyzt\ldots=s$; then $\Sigma s=0$.

(2) Prove also that the number of such solutions for which s exceeds zero is the coefficient of θ^n in the development of

$$\Sigma_{n=0}^{n=\infty}(-)^n\theta^{\frac{1}{2}(n^2+n)}+\Sigma_{n=-\infty}^{n=+\infty}(-)^n\theta^{\frac{1}{2}(3n^2+n)};$$

and that the number of solutions for which s equals zero is the coefficient of θ^n in the development of

$$(\theta-\theta^3)(1-\theta)+(\theta^6-\theta^{10})(1-\theta)(1-\theta^2)+\ldots\div\Sigma_{n=-\infty}^{n=+\infty}(-)^n\theta^{\frac{1}{2}(3n^2+n)}.$$

[As an example of (1), let $x+2y+3z+4t=4$; then the complete system of solutions is as follows:—

$x=0,\quad y=0,\quad z=0,\quad t=0;\qquad x=2,\quad y=1,\quad z=0,\quad t=0;$

$x=0,\quad y=0,\quad z=0,\quad t=1;\qquad x=4,\quad y=0,\quad z=0,\quad t=0;$

$x=1,\quad y=0,\quad z=1,\quad t=0;$

and the corresponding values of s are 1, 1, 0, 1, -3, whose sum is zero.]

3513. (Rev. C. Taylor, D.D.)—In the right circular cone, if B, L be extremities of the minor axis and latus rectum of an elliptic section, prove that the generating lines through B, L make angles θ, ϕ with the tangents at those points such that $\cos\theta=\cot\phi$.

3516. (M. Collins, B.A.)—The triangle ABD has a right angle A; in the straight line AB, take AB = BC = CD, and each = BA′ (a portion of BD); bisect CD in E, and let EA′ meet AD in F; then prove that $\angle$ ADF will be very little less than one-third of ABD.

3520. (J. W. L. Glaisher, B.A.)—Prove that

$$\left(\frac{d}{dq}\right)^{2i}e^{q^2/p^2}=p\left(-\frac{2d}{p\,dp}\right)^i\frac{e^{q^2/p^2}}{p}.$$

3525. (Walter Siverly.)—Find (1) the maximum ellipse inscribed between the major axis of an ellipse, any ordinate to it, and the curve; and (2) the locus of the centres of all such maximum ellipses that can be thus inscribed.

3527. (Professor Hudson, M.A.)—If three liquids which do not mix, and whose densities are ρ_1, ρ_2, ρ_3, fill a circular tube in a vertical plane, and if α, β, γ are the angles which the radii to the common surfaces make with the vertical diameter measured in the same direction, prove that

$$\rho_1(\cos\beta-\cos\gamma)+\rho_2(\cos\gamma-\cos\alpha)+\rho_3(\cos\alpha-\cos\beta)=0.$$

If there are equal quantities of each fluid, and if, in addition, the weights on each side of the vertical diameter are equal, obtain an equation to determine α which refers to the highest point of junction. Show that it is satisfied by $\alpha=30°$, and that therefore the densities are in arithmetical progression.

3530. (J. F. Moulton, M.A.)—Show that, if two families included under the functional equation $f(x^2-y^2,\ y^2-z^2)=0$ cut everywhere at right angles, the lines of intersection are lines of curvature on each.

3552. (H. MacColl, B.A.)—A certain mathematician solved a question in probability, and obtained q as his expression for the required chance. Not feeling satisfied, however, that this result was correct, he applied the test of experiment, and found that the event in question happened m times in n trials. Supposing P to be a fair estimate of the probability, independently of the experiment, that the result q was correct; show that, if the experiment be taken into account, the proper estimate is

$$a \div (a+b), \text{ in which } a = \mathrm{P}q^m (1-q)^{n-m}, \text{ and } b = \frac{m!\,(n-m)!}{(n+1)!}(1-\mathrm{P}).$$

[The numerical result can be easily calculated by the aid of STIRLING's theorem.]

3570. (Editor.)—Eliminate (1) α, β from

$$\frac{a^2x}{\alpha}-\frac{b^2y}{\beta} = 2a^2e^2, \quad \frac{x}{\alpha}-\frac{y}{\beta} = \frac{e^2}{b^2}(\alpha^2+\beta^2), \quad \frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2} = 1;$$

(2), ξ, v from $\mathrm{U} \equiv \dfrac{\xi^2}{a^2}+\dfrac{v^2}{b^2}-(\xi^2+v^2)^2 = 0, \quad \dfrac{d\mathrm{U}}{xd\xi} = \dfrac{d\mathrm{U}}{ydv} = \dfrac{\xi d\mathrm{U}}{d\xi}+\dfrac{vd\mathrm{U}}{dv}$;

and show that the results are identical.

3573. (J. Griffiths, M.A.)—Show that the conics

$$x^2+y^2+z^2-(l_1x+m_1y+n_1z)^2 = 0, \quad x^2+y^2+z^2-(l_2x+m_2y+n_2z)^2 = 0$$

(where $z = 0$ is the line infinity, and x, y are rectangular axes) will intersect at an angle θ if

$$\frac{l_1l_2+m_1m_2+n_1n_2-1}{\sqrt{(1-l_1^2-m_1^2-n_1^2)}\sqrt{(1-l_2^2-m_2^2-n_2^2)}} = \cos\theta.$$

3578. (Professor Evans, M.A.)—An urn contains m white and n black balls; two players A and B choose their colours and agree to play as follows. A ball drawn from the urn at random, and replaced before the commencement of the game, decides by its colour which player is to have the first turn; during the game the balls are to be drawn at random, and each ball drawn must be replaced in the urn before another is drawn; each player *gains a point* when he draws a ball of the colour chosen by himself, but *loses his turn* when he draws a ball of the opposite colour. A chooses the white; what is the probability that he will gain N points before B gains one point?

3582. (Professor Hudson, M.A.)—A uniform sphere is dragged by a horizontal force along a homogeneous fluid of twice its density; find the velocity which must be kept up in order that one-fourth of the vertical diameter may be immersed.

3583. (Artemas Martin, LL.D.)—What is the probability that a random shot will hit a target a feet square at a distance of b feet?

3586. (Professor Minchin, M.A.)—If a curve and its inverse be described with the same law of force, viz., μr^5p^{-5}, prove that both curves are inluded in the class whose equatiod is $r^6+ap^3+b = 0$.

3592. (Dr. Hirst, F.R.S.)—Let P be the common centre of three homographic pencils of rays possessing two triple lines (two lines with each of which three corresponding rays coincide). Then if, on a fixed

conic through P, two fixed points Q and R and a variable point M be taken, the envelope of a conic which is inscribed in the triangle MQR so as to touch the two rays corresponding to MP is a cubic which has a double point at P, and passes through Q and R as well as through the intersections of the fixed conic with the triple line.

3627. (Dr. Hirst, F.R.S.)—P, Q, R, S being the four intersections of two fixed conics Σ and Σ', and T_1, T_2 any other fixed points on Σ'; if a variable line through P be drawn to intersect Σ in M and Σ' in M′, the conic inscribed in the triangle MQR so as to touch the connectors of M′ with T_1 and T_2 will envelope a cubic which has a double point at S and likewise passes through Q, R, T_1, and T_2. [Examine the special case where Q and R are the circular points at infinity.]

3631. (Editor.)—If two diagonals of a regular polygon are drawn at random, find the probability that they will intersect within the figure. Again, if three diagonals are drawn at random, find the respective probabilities of 0, 1, 2, 3 intersections.

3634. (J. J. Walker, M.A.)—An arc of rigid uniform circular hoop is to be placed in a position of equilibrium, with its convex side resting on two horizontal pins fixed in a vertical wall. If there is no friction, show that, for such a position to be possible, the inclination (to the horizontal) of the line joining the pins must not be greater than the less of the two angles ABC, BAC; AB being the chord of the arc, and AC the chord, equal in length to that line, of a portion of the arc. Determine the pressures when this condition is satisfied.

3637. (G. S. Carr.)—An ironclad ship, sailing N.W. at the rate of 15 miles an hour, is three-quarters of a mile N. of a battery on the shore, from which it is required to throw a shell so that it may *fall* upon the deck of the vessel. Required the angles (true to 10 seconds) for pointing the mortar; the velocity of the discharge being 644 feet per second, the resistance of the air being neglected, and the force of gravity taken as 32·2 feet per second.

3642. (T. Cotterill, M.A.)—(1) What is the locus of the cusps of a system of parallel curves? (2) Show that, in the neighbourhood of a cusp on a curve or its evolute, we can render visible two cusps and a node of a parallel to the curve by taking an appropriate modulus. [Mr. Cotterill remarks that Zeuthen has given $m(m+2n-5)+\tau$ as the order of the locus of the nodes of the system of parallels to a curve of the order m, class n, with τ bitangents.]

3655. (Rev. Dr. Booth, F.R.S.)—De Moivre's theorem is in circular trigonometry $\quad (\cos\phi + i\sin\phi)^n = \cos n\phi + i\sin n\phi$;
show that in parabolic trigonometry

$$\begin{aligned}(\sec\phi + \tan\phi)^n = {} & \sec(\phi \perp \phi \perp \phi \text{ to } n \text{ terms}),\\ & + \tan(\phi \perp \phi \perp \phi \text{ to } n \text{ terms});\end{aligned}$$

where $\perp$ and $\top$ may be called parabolic plus and minus, and their meaning given by definition.

3660. (Rev. A. F. Torry, M.A.)—Two particles describe the same

ellipse subject to the same force in the centre: show that their directions of motion at any time intersect on a similar ellipse.

3664. (A. Martin.)—A radius is drawn at random in a semicircle, and a circle inscribed in each sector; find the average distance between the centres of the inscribed circles.

3669. (From Whitworth's "Choice and Chance.")—A vessel is filled with three liquids whose specific gravities in descending order of magnitude are S_1, S_2, S_3. All volumes of the several liquids being equally likely; prove that the chance of the specific gravity of the mixture being greater than S is

$$\frac{(S_1-S)^2}{(S_1-S_2)(S_1-S_3)}, \quad \text{or} \quad \frac{(S-S_3)^2}{(S_2-S_3)(S_1-S_3)},$$

according as S lies between S_1 and S_2, or between S_2 and S_3.

3671. (Professor Hudson, M.A.)—Suppose that the prosperity of a country varies as the excess of the average individual wealth over a given sum. By defeat and civil war one-mth of the wealth and one-nth of the population are destroyed, and the nation has to pay a property tax of $100r$ per cent. to the victor: find what proportion of the population the ruling government must now put to death in order that the prosperity may be the same as before. [Professor Hudson states that this question was included amogst those which, as Junior Moderator, he gave for a Cambridge Tripos, but that, at the earnest request of his colleagues, it was cut out for sentimental reasons.]

3676. (Professor Sylvester.)—If $a+2b+3c+\ldots+rl=n$, prove that on making $\phi x = x^r - x^{r-1} - x^{r-2} \ldots - 1 = 0$,

$$\Sigma \frac{\Pi n}{\Pi a \Pi b \Pi c \ldots \Pi l} = \Sigma \frac{x^{n+r-1}}{\phi' x}.$$

3679. (Professor Sir R. E. Ball, F.R.S.)—If a rigid body having three degrees of freedom be in equilibrium under the action of gravity, the restraints must be such as would permit the body to be rotated about an axis passing through the centre of gravity.

3681. (Dr. Hirst, F.R.S.)—Let L, M, N and Q, R, S be two sets of fixed collinear points on a cubic which has a double point P; and let A, B be a (variable) pair of conjugate points of a given involution on this cubic. If AB cut the curve again in C, CQ in P′, and CR in P″, prove that the quartic which has double points at A and B, touches each of the lines PP′, PP″ at P, and passes through L, M, and N necessarily passes also through S, and moreover envelopes a conic.

3684. (T. Cotterill, M.A.)—In a plane take n points and connect them by lines so as to form a polygon of n sides. The polars of the points to a conic form a fresh polygon with n sides corresponding to the n points. Show that the envelope curve determined by the $\frac{1}{2}n(n-3)$ lines in the first figure connecting points not already joined is the reciprocal polar of the locus curve determined by the $\frac{1}{2}n(n-3)$ intersections of the non-consecutive sides of the second figure.

3695. (Artemas Martin, LL.D.)—Find the probability that a random shot will hit a circular target of a feet radius at a distance of b feet.

3700. (Rev. W. A. Whitworth, M.A.)—P, Q, R, S, ... are any number of points on a conic section. Prove that, if the ratios of the chords of

curvature at P and Q in direction PQ, at Q and R in direction QR, at R and S in direction RS, and so on round the conic, be compounded, the resulting ratio is a ratio of equality.

3705. (From Whitworth's "Choice and Chance.")—If n numbers be selected at random, what are the respective chances that their continued product in the common scale of notation will end with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9?

3706. (R. Tucker, M.A.)—If d, D be the diameters of the inscribed and circumscribed circles of a triangle, δ the distance between their centres, and h the harmonic mean between the radii of the two circles which can be described through the above centres touching a side of the triangle, then
$$\Sigma\left(\frac{1}{h-d}\right)=\frac{4d+D}{\delta^2-d^2}.$$

3708. (Professor Sylvester.)—A trifurcal ramification is a system of things or ideas, say points, consisting of terminals and joints, so conjoined, say by lines, that on taking any of the former as a root, the remaining terminals and joints are related to it and to each other after the manner of a florescence, in which, at each joint, the line, say branch, which leads up to it divides into two. Prove that the possible schemes of trifurcal ramification with 4, 6 and 8 joints, may be obtained by cutting 3, 4, and 5 edges of a pyramid, a wedge and a cube, or truncated wedge respectively; no regard in any ramification scheme being had to the directions or lengths of the branches. Prove also that all the possible schemes of quadrifurcal aborescence (that, by the way, of the hydrocarbon series) may be obtained by 5 sections of the edges of an octahedron. Prove furthermore, for each of the cases above referred to, that all the trifurcal and quadrifurcal ramifications, with an inferior or intermediate number of joints, may be obtained by the process of cutting the edges as before combined with that of freeing them from their connexion with each other at the angles of the generating figure.

3709. (Professor Cayley, F.R.S.)—Mention what form of given relation $\phi(a, b, c, \ldots)=0$ between the roots of a given equation will in general serve for the rational determination of the roots; explain the case of failure: and state what information as to the roots is furnished by a given relation not of the form in question.

3715. (Editor.)—A point is taken at random within a circle, and a chord is drawn at right angles to the straight line joining the point with the centre. Find the average of the ratio of the parts into which (1) the circumference, and (2) the circle, is divided by the chord; also the probability that either ratio will be less than a given magnitude.

3719. (Dr. Hirst, F.R.S.)—Let A′, B′, C′ be the intersections of a given line t with the sides, respectively opposite to the vertices A, B, C of a triangle inscribed in a conic Σ. If through the intersections B′, C′, the vertex A, and any fixed point D in the plane, a (variable) conic be described, cutting Σ again in P, Q, R, prove that the quartic curve which has double points at P, Q, R and passes through A′, B′, C′, as well as through any two fixed points E and F on the conic Σ, necessarily passes likewise through the intersection of t and EF, and moreover envelopes another conic.

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